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Scalar Waves In An Almost Cylindrical Spacetime

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Scalar Waves In An Almost Cylindrical Spacetime

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ABSTRACT

The scalar wave equation is investigated for a scalar field propagating in a spacetime background $ds^2 = e^{2a}(-dt^2 + dr^2) + R\left(e^{-2\psi}d\phi^2 + e^{2\psi}dz^2\right)$. The metric is compactified in the radial direction. The spacetime slices of constant ϕ and z are foliated into outgoing null hypersurfaces by the null coordinate transformation $u = t - r$. The scalar field imitates the amplitude behavior of a light ray, or a gravitational wave, traveling along a null hypersurface when the area function R is a constant or is a function of u. These choices for R restrict the gravitational wave factor ψ to being an arbitrary function of u.

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CONTENTS

1 Introduction

Gravitational waves, predicted by general relativity, add another layer of strangeness to our perception and understanding of nature. To know why gravitational waves are strange, we must understand the general relativistic description of gravity: gravity is the curvature of spacetime and vise versa. Based on this, gravitational waves are curvature propagating through spacetime [1, 2]. Gravitational waves are also ripples of spacetime, according to another description. In principle, it is possible to detect gravitational waves. As they pass through a material object, they induce a strain, which is a change in the length of an object divided by its length [3, Definition of strain]). However, detecting gravitational waves has been elusive due to the fact that gravity couples weakly to matter. The effort to directly detect gravitational waves was pioneered by Joseph Weber in the 1960's [4, 5, 6, 7]. Though he failed to detect gravitational waves, his work started the experimental field of gravitational wave detection. Since then, it has become a global effort. In the last twenty years, sensitive gravitational wave detectors known as laser interferometers, have been designed, constructed, and made operational. Major scientific collaborations operating these type of detectors are: LIGO in the USA, GEO600 in Germany, VIRGO in France and Italy, and TAMA300 in Japan [8, 9, 10, 11, 12]. Gravitational waves have only been detected indirectly. In 1993, the Noble prize in physics was awarded to Taylor and Hulse for the discovery of the first binary neutron star system, *PSR 1913 + 16*. The orbital period of this system decays through the emission of gravitational waves at a rate predicted by general relativity.

Sources of detectable gravitational waves are at very large distances from earth based detectors. Because of this and the weakness of the interaction between gravity and matter, gravitational wave amplitudes are extremely small. To visualize the size of these amplitudes, consider the following: even laser interferometer detectors, with 4 km arms and at strain sensitivity levels of 10^{-18}, have not detected any gravitational waves. If L is the length of an interferometer arm, the strain in that arm is given by the change in L divided by $L : \frac{\Delta L}{L}$. Because gravitational wave amplitude h is proportional to strain, a gravitational wave with amplitude of 10^{-18} can induce a length change of the order of magnitude of 10^{-15} meters in a 4 km interferometer arm. This length change is approximately 10,000 times smaller than the Bohr radius. Gravitational wave signals are buried in noise, but thanks to a technique called *matched filtering* [13], it is possible to extract them. The ideal is to integrate, in time, known waveforms over the data. Obtaining gravitational wave forms, however, presents another challenge. Waveforms are found by solving the equations of general relativity, which are the Einstein field equations, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$. Solving these equations is an arduous task, except for the weak field cases, where gravitational waves are perturbations of a flat background.

Together, the Einstein field equations form a coupled system of ten nonlinear, hyperbolic, partial differential equations. In their standard form, $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$, they are unsuitable for simulation on a computer. Because there is no preferred time direction, a problem is created, since time is unified with space to form spacetime. As a consequence, the four equations $G_{tt} = 8\pi T_{tt}, G_{tx^1} = 8\pi T_{tx^1}, G_{tx^2} = 8\pi T_{tx^2}$, and $G_{tx^3} = 8\pi T_{tx^3}$, in which x^1, x^2, and x^3 are space coordinates, contain no second time derivatives of the spacetime metric $g_{\alpha\beta}$. Instead of evolving the metric forward in time, these equations constrain the metric on a time slice. Therefore, to get a time evolution scheme, a time direction must be chosen. This is done by slicing spacetime into either space-like hypersurfaces, time-like hypersurfaces, light-like hypersurfaces, or some hybrid of the three. The Einstein field equations are then projected onto an initial slice. Then, the spacetime slice is evolved forward in time. These are some of the difficulties inherited by *numerical relativity* [14], the field that specializes in simulating the Einstein field equations.

When studying gravitational waves, or wave phenomena that travel at the speed of light, it is ideal to slice spacetime into light-like (null) hypersurfaces. Phenomena that travel at the speed of light, travel solely within null hypersurfaces all the way to infinity. Instead of truncating the waveform at some finite distance from the source, we are able to obtain a complete description of the wave from the source to infinity. Having complete confinement to a null hypersurface, we can map infinity to within a finite distance from the source by compactifying the spacetime. In this work, we compactify the radial coordinate of a spacetime with a two parameter space-like isometry group. Then, we slice the spacetime into null hypersurfaces for a specific class of observers. The scalar wave equation is then derived in this spacetime and parameters are chosen for the scalar field to imitate the amplitude behavior of a light ray traveling along a null hypersurface.

Section 2 introduces general relativity and its mathematical background. Section 3 covers basic mathematics and physics of gravitational waves in the weak field case. Section 4 introduces numerical relativity by slicing spacetime into space-like hypersurfaces and null hypersurfaces. In section 5, we slice a spacetime with a two parameter space-like isometry group into null hypersurfaces for a specific class of observers. Then, we derive and manipulate the scalar wave equation to have the scalar field imitate light rays traveling along null hypersurfaces.

2 Introduction to General Relativity

Each topic discussed in this section can be found in the following general relativity and differential geometry texts [15, 16, 17].

2.1 Mathematics

2.1.1 Manifolds

As humans, we have an intuitive grasp of what a manifold is. Examples of manifolds are: the surface of a table, a globe, and a cylinder; an idealized straight line from Euclidean geometry; and three-dimensional Euclidean space. The points on manifolds are described by overlapping coordinate systems to prevent coordinate singularities from raising up. Coordinate systems and manifolds are two distinct entities. Consider three-dimensional Euclidean space as an illustration. It is intrinsically flat, but if it is described using spherical coordinates, it appears to be curved. The apparent curvature, however, is due to the coordinates and not to the underlying properties of the manifold.

Transitioning to a more mathematical perspective, a set M is an *n-dimensional real manifold* [17, pg. 23], if there exists about each point P of M a neighborhood U that has a continuous bijective map, $\phi : U- > V$, into some open set V of Euclidean n-space (R^n) . From this requirement, we see that a point P of M is associated with an n-tuple $(y^1(P), ..., y^n(P))$.Since the map ϕ is a bijection, the neighborhood U of point P acquires a coordinate system under ϕ. A neighborhood U with map ϕ is called a *chart* and is denoted by (U, ϕ). An important property of M is determined by the mappings between charts. Because each point of M is in at least one neighborhood, we have points whose neighborhoods overlap. If the neighborhood U of point P overlaps the neighborhood H of point S, where the map ψ maps H into an open set of R^n, (not necessarily the same open set that ϕ maps U into) then the overlap region $U \cap H$ is assigned two coordinate systems. The two coordinate systems can be related by the following steps. Since ψ is required to be a bijection, the inverse mapping $\psi^{-1} : \psi(U \cap H) -> U \cap H$ maps an n-tuple $(y^1(T), ..., y^n(T))$ to the point T in the intersection of U and H. Next, the map $\phi : U \cap H -> \phi(U \cap H)$ maps the point T into the n-tuple $(z^1(T), ..., z^n(T))$. Composing the maps ψ^{-1} and ϕ yields the mapping $\phi \circ \psi^{-1} : \psi(U \cap H) -> \phi(U \cap H)$. This implies that the n-tuples have the following relationship:

$$
\begin{aligned}
z^1(T) &= z^1(y^1(T), ..., y^n(T)) \qquad (2.1)\\
&.\\
&.\\
&.\\
z^n(T) &= z^n(y^1(T), ..., y^n(T)).
\end{aligned}
$$

The mapping $\phi \circ \psi^{-1} : \psi(U \cap H) -> \phi(U \cap H)$ and the functional relationship in equation (1) is called a *coordinate transformation*. If the functions $z^1, ..., z^n$ are differentiable up to order n, including the n^{th} order, then the charts (U, ϕ) and (H, ψ) are said to be C^n related. If the set of charts (U_i, ϕ_i), $i = 1, .., k$ that covers the manifold are all C^n related, then the manifold is called a C^n *manifold*. As long as $n \geq 1$ we have a differentiable manifold, which is what we will be working with. In this situation, it is useful to consider vectors as directional derivative operators. Directional derivatives and vectors are isomorphic. Hence,

$$v = \partial_v = \left(\frac{d}{d\tau}\right)_{along\ curve}, \tag{2.2}$$

meaning that v is a tangent vector to some curve of parameter τ [15, pg. 227, eq. (9.1)]. If f is a curve in an n-dimensional manifold M, with parameter τ, then the tangent vector v along this curve is

$$\begin{aligned} v &= \frac{df}{d\tau} = \sum_{k=1}^{n} \frac{\partial f}{\partial y^k}\frac{dy^k}{d\tau} = \sum_{k=1}^{n} \frac{dy^k}{d\tau}\frac{\partial f}{\partial y^k} \\ &= \left(\sum_{k=1}^{n} \frac{dy^k}{d\tau}\frac{\partial}{\partial y^k}\right) f = \left(\sum_{k=1}^{n} v^k e_k\right) f, \end{aligned} \tag{2.3}$$

where $v^k = \frac{dy^k}{d\tau}$, and $e_k = \frac{\partial}{\partial y^k}$. Basis vectors represented as partial derivatives are called *a coordinate basis* [15, section 9.3]. The vector v can be written as $v = \sum_{k=1}^{n} \frac{dy^k}{d\tau}\frac{\partial}{\partial y^k}$, when it is understood that v is tangent to some curve f.

2.1.2 Tensors

Tensors [17, pg. 57] are a generalization of linear functions. A key concept to understanding tensors is that they accept *arguments*, elements of vector spaces, and then map these arguments into a real number. This number could be a complex number, but assume we are working with real valued tensors. A tensor can take any number of arguments. If n is the number of arguments a tensor can take, then the tensor is said to be of rank n. The general properties of a second rank tensor L (which applies equally to an n$^{\text{th}}$ ranked tensor), with ω a real number, are the following:

$$L(\omega x_1, x_2) = \omega L(x_1, x_2), \quad L(x_1, \omega x_2) = \omega L(x_1, x_2) \tag{2.4}$$

$$L(x_1 + y_1, x_2) = L(x_1, x_2) + L(y_1, x_2), \quad L(x_1, x_2 + y_2) = L(x_1, x_2) + L(x_1, y_2).$$

Tensors abound in mathematics. All linear functions of the form $f(x) = mx$ are tensors. Row vectors are tensors because they map column vectors to a real number. Definite integrals $\int_a^b f(x)dx$ are tensors, since the real continuous function $f(x)$ is mapped to a real number. The dot product $A \cdot B = A_1B_1 + ... + A_nB_n = |A||B|\cos\theta$, $A, B \in R^n$ is a tensor too, since we can view the vector A as mapping the vector B to the real number $A \cdot B$.

Linear Forms *Linear forms* [17, pg. 49] are real valued linear functions defined on a vector space V. Letting ω be a form, we have $\omega : V- > R$, or $\omega(v) = m$ with $v \in V$ and $m \in R$. The set of forms over V is denoted by $V^{'}$. It is a vector space and is called *the space dual to V*. To develop the basic properties of linear forms, let V be an n dimensional vector space. Hence, a vector $v \in V$ is expandable in terms of n linearly independent basis vectors $e_1, ..., e_n$:

$$v = v^1 e_1 + ... + v^n e_n. \tag{2.5}$$

In terms of row and column vectors, we can write the vector v as

$$v = \begin{bmatrix} v^1 & ... & v^n \end{bmatrix} \begin{bmatrix} e_1 \\ . \\ . \\ . \\ e_n \end{bmatrix}. \tag{2.6}$$

Denoting the column vector of n linearly independent basis vectors by e, we have a bijective map $e : R^n -> V$, which takes the n-tuple $(v^1, .., v^n) \in R^n$ into a vector $v \in V$. Because e is a bijective map its inverse, $e^{-1} : V -> R^n$, is also a bijective map. This means the inverse map e^{-1} is the array of expansion coefficients of v, meaning $e^{-1}(v) = (v^1, .., v^n)$. Thus, each coefficient v^r of the vector v can be written as

$$v^r = e^{-1}(v) \begin{bmatrix} \delta_r^1 \\ . \\ . \\ . \\ \delta_r^n \end{bmatrix}, \tag{2.7}$$

where the δ_i^n are *Kronecker deltas*. The relation $v^r = e^{-1}(v)\delta_r^i$ is the linear mapping $v^r : V -> R$. The coefficients v^r are linear forms. In more explicit notation, letting β^r be a linear form, we have $\beta^r(v) = v^r$.Now let a form ω act on the vector v :

$$\omega(v) = \omega(v^1 e_1 + ... + v^n e_n). \tag{2.8}$$

Because ω is a linear map, we have

$$\omega(v^1 e_1 + ... + v^n e_n) = v^1 \omega(e_1) + ... + v^n \omega(e_n). \tag{2.9}$$

This means that

$$\omega(v)=\sum_{j=1}^{n}v^j\omega(e_j)=\sum_{j=1}^{n}\omega_j v^j=\sum_{j=1}^{n}\omega_j\beta^j(v), \tag{2.10}$$

where $\omega(e_j)$ is the real number ω_j. Hence, we have an expansion of an arbitrary linear form ω, where the $\beta^j(v)$ are identified as the basis of ω, and the ω_j are the expansion coefficients. The linear forms $\beta^j(v)$ form a basis for the dual space $V^{'}$ and are called *the basis dual to e.* To obtain a useful relationship between the basis forms and the vector basis, consider the expansion of an arbitrary basis vector e_j :

$$e_j=\sum_{k=1}^{n}v^k e_k. \tag{2.11}$$

Because vector coefficients are the linear forms $\beta^k(e_j)=v^k$, we have

$$e_j=\sum_{k=1}^{n}\beta^k(e_j)e_k. \tag{2.12}$$

To preserve the linear independence of the arbitrary basis vector, it is required that

$$\beta^k(e_j)=\delta_j^k=\begin{cases}\delta_j^k=0, k\neq j\\ \delta_j^k=1, k=j\end{cases}. \tag{2.13}$$

In summary, using the differential geometry notation $h(k)=h\cdot k=k\cdot h=k(h)$, the relations developed between vectors and linear forms are the following:

$$\beta^k(e_j) = \beta^k\cdot e_j=e_j(\beta^k)=\delta_j^k \tag{2.14(a)}$$

$$\omega(e_j) = \omega\cdot e_j=e_j(\omega)=\omega_j \tag{2.14(b)}$$

$$\beta^j(v) = \beta^j\cdot v=v(\beta^j)=v^j. \tag{2.14(c)}$$

Now, consider the space V'', which is the dual of the dual space V'. Elements in V'' are linear functions that map forms to numbers. For some $h' \in V''$ we have $h' : V' -> R$, or $h'(\alpha) = m$ for $\alpha \in V'$ and $m \in R$. However, instead of defining a new object that maps forms into numbers, we can simplify our situation by identifying the number $\alpha(v)$, $v \in V$ with each form α. This means that

$$h'(\alpha) = \alpha(v). \tag{2.15}$$

We can view $\alpha(v)$ not only as a form mapping a vector to a number, but also as a vector mapping a form to a number. This latter view becomes obvious if the vector remains the same and the form is changed. Because of this dual view, h' and a vector v can be identified as the same object. Thus, we can write

$$h'(\alpha) = v(\alpha) = \alpha(v). \tag{2.16}$$

More Tensors By the definition given of tensors, we understand that a form α and a vector v are both first rank tensors. A tensor of rank n, where k arguments are vectors, l arguments are forms, and $k + l = n$, is called a k^{th} rank covariant l^{th} rank contravariant tensor. It can also be called an n^{th} rank mixed tensor. Therefore, a *form* is a first rank covariant tensor and a *vector* is a first rank contravariant tensor. Higher rank tensors can be formed from α and v by the tensor product operator $\otimes$. A second rank tensor that can be formed using α and v is

$$\alpha \otimes v(u, \beta) = \alpha(u)v(\beta). \tag{2.17}$$

Likewise, we can form a fourth rank tensor

$$\alpha \otimes v \otimes \alpha \otimes v(u, \beta, m, \omega) = \alpha(u)v(\beta)\alpha(m)v(\omega). \tag{2.18}$$

A general example is the tensor product of $Y(\alpha_1, ..., \alpha_n, v_1, ..., v_m)$ and

$W(\beta_1, ..., \beta_l, u_1, ..., u_k)$:

$$\begin{aligned} & Y \otimes W(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_l, v_1, ..., v_m, u_1, ..., u_k) \\ = \;& Y(\alpha_1, ..., \alpha_n, v_1, ..., v_m)W(\beta_1, ..., \beta_l, u_1, ..., u_k). \end{aligned} \tag{2.19}$$

The resulting tensor is a $(m + k)^{th}$ rank covariant, $(n + l)^{th}$ rank contravariant tensor. Now consider a tensor of the form

$$L(\alpha, \gamma, u, v), \tag{2.20}$$

where $u, v \in V$ and $\alpha, \gamma \in V'$. This tensor maps a pair of vectors in V and a pair of forms in V' to a real number $m \in R$. This means that $L : V \times V \times V' \times V' -> R$. Now, insert the expansions of vectors u, v, and linear forms α and γ into L :

$$L(\alpha, \gamma, u, v) = L(\sum_{k=1}^{n} \alpha_k \beta^k, \sum_{l=1}^{n} \gamma_l \beta^l, \sum_{i=1}^{n} u^i e_i, \sum_{j=1}^{n} v^j e_j). \tag{2.21}$$

Because of linearity, we obtain

$$L(\alpha, \gamma, u, v) = L(\sum_{k=1}^{n} \alpha_k \beta^k, \sum_{l=1}^{n} \gamma_l \beta^l, \sum_{i=1}^{n} u^i e_i, \sum_{j=1}^{n} v^j e_j) = \sum_{k,l,i,j=1}^{n} \alpha_k \gamma_l u^i v^j L(\beta^k, \beta^l, e_i, e_j). \tag{2.22}$$

To simplify notation, the summation symbol will be dropped when summing over repeated indices when one index is down and the other index is up. Form α is written as

$$\alpha = \sum_{k=1}^{n} \alpha_k \beta^k = \alpha_k \beta^k, \tag{2.23}$$

the vector v is written as

$$v = \sum_{j=1}^{n} v^j e_j = v^j e_j, \tag{2.24}$$

and the tensor L is written as

$$L(\alpha, \gamma, u, v) = \sum_{k,l,i,j=1}^{n} \alpha_k \gamma_l u^i v^j L(\beta^k, \beta^l, e_i, e_j) = \alpha_k \gamma_l u^i v^j L(\beta^k, \beta^l, e_i, e_j). \tag{2.25}$$

Now, the numbers $L(\beta^k, \beta^l, e_i, e_j)$ are the tensor components

$$L(\beta^k, \beta^l, e_i, e_j) = L^{kl}{}_{ij}. \tag{2.26}$$

Notice that the components k and l are a mapping of the forms β^k and β^l, and that components i and j are a mapping of the vectors e_i and e_j. This suggests that the basis of L is

$$e_k \otimes e_l \otimes \beta^i \otimes \beta^j. \tag{2.27}$$

Since this is the basis of L, then L can be written as

$$L = L^{kl}{}_{ij} e_k \otimes e_l \otimes \beta^i \otimes \beta^j. \tag{2.28}$$

To see that this is the correct basis for L, evaluate $L(\alpha, \gamma, u, v)$:

$$
\begin{aligned}
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} e_k \otimes e_l \otimes \beta^i \otimes \beta^j (\alpha,\gamma,u,v) \qquad (2.29)\\
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} e_k(\alpha) e_l(\gamma) \beta^i(u) \beta^j(v)\\
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} e_k(\alpha_h \beta^h) e_l(\gamma_m \beta^m) \beta^i(u^p e_p) \beta^j(v^s e_s)\\
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} \alpha_h e_k(\beta^h) \gamma_m e_l(\beta^m) u^p \beta^i(e_p) v^s \beta^j(e_s)\\
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} \alpha_h \gamma_m u^p v^s \delta^h_k \delta^m_l \delta^i_p \delta^j_s\\
L(\alpha,\gamma,u,v) &= L^{kl}{}_{ij} \alpha_k \gamma_l u^i v^j = \alpha_k \gamma_l u^i v^j L^{kl}{}_{ij}.
\end{aligned}
$$

In general, a tensor T has components $T^{k_1 \ldots k_n}{}_{l_1 \ldots l_m}$ and basis $e_{k_1} \otimes \ldots \otimes e_{k_n} \otimes \beta^{l_1} \otimes \ldots \otimes \beta^{l_m}$:

$$T = T^{k_1 \ldots k_n}{}_{l_1 \ldots l_m} e_{k_1} \otimes \ldots \otimes e_{k_n} \otimes \beta^{l_1} \otimes \ldots \otimes \beta^{l_m}. \qquad (2.30)$$

Now, the tensor L can either be symmetric or antisymmetric. It is symmetric if

$$L(\alpha, \gamma, u, v) = L(\gamma, \alpha, u, v) \qquad (2.31)$$

and

$$L(\alpha, \gamma, u, v) = L(\alpha, \gamma, v, u). \qquad (2.32)$$

It is antisymmetric if

$$L(\alpha, \gamma, u, v) = -L(\gamma, \alpha, u, v), \qquad (2.33)$$

and

$$L(\alpha, \gamma, u, v) = -L(\alpha, \gamma, v, u). \qquad (2.34)$$

The tensor L could be symmetric with respect to the form arguments and antisymmetric with respect to the vector arguments. Likewise, L could be antisymmetric with respect to the form arguments and symmetric with respect to the vector arguments. These results apply equally to the general tensor T, where T could be totally symmetric or antisymmetric or both symmetric and antisymmetric in its arguments.

Metric Tensor An important tensor in general relativity is a symmetric second rank covariant tensor g, also known as the *metric tensor* [17, pgs. 64-70]. The metric tensor allows one to compute a scalar product for vectors in an $n-$dimensional space. This scalar product is analogous to the ordinary dot product of three-dimensional Euclidean space. If the metric tensor of an n-dimensional space is known, the scalar product of two vectors, u and v, is

$$u \cdot v = g(u, v). \tag{2.35}$$

If u and v are expandable in terms of a basis set $\{e_i, i = 1, ..., n\}$, then

$$u \cdot v = g(u, v) = g(u^i e_i, v^j e_j) = g(e_i, e_j) u^i v^j = g_{ij} u^i v^j, \tag{2.36}$$

where g_{ij} are the coefficients of the metric tensor. Since the metric tensor is a scalar product of vectors, the metric coefficients are the scalar product of basis vectors:

$$e_i \cdot e_j = g(e_i, e_j) = g_{ij.} \tag{2.37}$$

In three-dimensional Euclidean space, using Cartesian coordinates, the matrix of scalar products of basis vectors is

$$e_i \cdot e_j = g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.38}$$

Transforming to spherical coordinates, we have

$$e_i \cdot e_j = g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{2.39}$$

The *line element* of a space is found by inserting the infinitesimal displacement vector $dx = dx^i e_i$ into both slots of the metric tensor:

$$ds^2 = g(dx, dx) = g(dx^i e_i, dx^j e_j) = g(e_i, e_j)dx^i dx^j = g_{ij}dx^i dx^j. \tag{2.40}$$

In three-dimensional Euclidean space, using Cartesian, and then spherical coordinates, we have the familiar results

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2.41}$$

and

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.42}$$

The metric tensor has an inverse g^{-1}. If g is the scalar product of two vectors, then g^{-1} is the scalar product of two linear forms. Letting α and ω be linear forms,

$$\alpha \cdot \omega = g^{-1}(\alpha, \omega). \tag{2.43}$$

If the linear forms α and β are expandable in terms of a basis set $\{\beta^i, i = 1, ..., n\}$, then

$$\alpha \cdot \omega = g^{-1}(\alpha, \omega) = g^{-1}(\alpha_l \beta^l, \omega_j \beta^j) = g^{-1}(\beta^l, \beta^j)\alpha_l \omega_j = g^{lj}\alpha_l \omega_j. \tag{2.44}$$

The relationship between the coefficients g_{ij} and g^{lj} is

$$g_{ij}g^{lj} = \delta^l_j. \tag{2.45}$$

It is useful to think of the metric and its inverse as maps. Suppose that only one vector is inserted into the metric tensor g. Letting β^i and β^j be basis forms, yields

$$\begin{aligned} g(u,\ ) &= g_{ij}\beta^i \otimes \beta^j(u,\ ) = g_{ij}\beta^i \otimes \beta^j(u^k e_k,\ ) \qquad (2.46)\\ &= g_{ij}u^k \delta^i_k \beta^j(\ ) = g_{ij}u^i \beta^j(\ ). \end{aligned}$$

If $g_{ij}u^i\beta^j(\ )$ acts on a vector $v = v^l e_l$, then

$$g_{ij}u^i\beta^j(v) = g_{ij}u^i v^j = g(u, v). \tag{2.47}$$

The metric tensor g maps the vector u into the linear form $g(u,\ )$. The components of this linear form are

$$g_{ij}u^i = u_j. \tag{2.48}$$

Because the metric tensor is invertible, it is easily found that

$$u^i = g^{ij}u_j. \tag{2.49}$$

These equations imply that the metric tensor can be used to raise and lower indices on tensors. Insert one linear form into the inverse metric tensor g^{-1}. It yields

$$\begin{aligned} g^{-1}(\alpha, \ ) &= g^{ij} e_i \otimes e_j(\alpha, \ ) = g^{ij} e_i \otimes e_j(\alpha_k \beta^k, \ ) \\ &= g^{ij} \alpha_k \delta_i^k e_j(\ ) = g^{ij} \alpha_i e_j(\ ). \end{aligned} \tag{2.50}$$

Letting $g^{ij}\alpha_i e_j(\ )$ act on the linear form $\omega = \omega_k \beta^k$, leads to

$$g^{ij}\alpha_i e_j(\omega) = g^{ij}\alpha_i \omega_j = g^{-1}(\alpha, \omega). \tag{2.51}$$

Therefore, the inverse metric tensor g^{-1} maps the linear form α into the vector $g^{-1}(\alpha, \ )$. In general relativity, we will be working with four-dimensional manifolds known as *spacetimes*. The metrics of these manifolds will have the signature $(-, +, +, +)$. Lowercase greek letters will denote the spacetime index $0, ..., 3$. If $g_{\alpha\beta}$ are the coefficients of a spacetime metric g, then it may be written as the 4×4 matrix:

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{2.52}$$

A specific spacetime metric is the *Minkowski metric* [17, pg. 71] of special relativity:

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.53}$$

A property of spacetime metrics, unlike other metrics, is that a vector can have zero length without being the zero vector. To illustrate, let v be a vector in

Minkowski space. Its length is

$$|v| = \sqrt{\eta(v,v)} = \sqrt{\eta_{\alpha\beta}v^\alpha v^\beta} = \sqrt{-(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2}. \qquad (2.54)$$

If $(v^3)^2 = (v^2)^2 = (v^1)^2 = \frac{1}{3}(v^0)^2$, then $|v| = 0$. All vectors of length zero are said to be light-like (light-like vectors are also called null vectors). Besides being light-like, vectors can be in either of two other categories. A vector v is said to be space-like if $v^2 > 0$. It is said to be time-like if $v^2 < 0$. Thus, vectors in a spacetime are partitioned into three distinct groups. These groups give the causal structure of every spacetime. The term *causal* is used because all events joined by light-like vectors are connected by signals propagating at the speed of light. All events joined by time-like vectors are connected by signals propagating at the speed of light, or less, and all events joined by space-like vectors are connected by signals that travel faster than the speed of light. Since no physical signal can travel faster than the speed of light, only events joined by light-like and time-like vectors are causally connected.

2.1.3 Curvature

Covariant Derivatives Consider a vector $v = v^k e_k$ in an n-dimensional space. Differentiating v with respect to a parameter λ, yields

$$\frac{dv}{d\lambda} = \frac{dv^k}{d\lambda}e_k + v^k\frac{de_k}{d\lambda} = \frac{\partial v^k}{\partial x^i}\frac{dx^i}{d\lambda}e_k + v^k\frac{\partial e_k}{\partial x^i}\frac{dx^i}{d\lambda}. \qquad (2.55)$$

If the basis vectors do not shift or turn, such as Cartesian basis vectors, then $\frac{\partial e_k}{\partial x^i} = 0$. However, if the basis vectors do shift or turn, such as spherical basis vectors, then

$$\frac{\partial e_k}{\partial x^i} = \Gamma^j{}_{ki} e_j. \tag{2.56}$$

The quantities $\Gamma^j{}_{ki}$, known as connection coefficients, represent how much the basis vector e_k has shifted. Substituting this quantity into the total derivative of v, yields

$$\frac{dv}{d\lambda} = \frac{\partial v^k}{\partial x^i}\frac{dx^i}{d\lambda} e_k + v^k \Gamma^j{}_{ki} e_j \frac{dx^i}{d\lambda}. \tag{2.57}$$

Renaming two of the contracted sets of indices gives

$$\begin{aligned} \frac{dv}{d\lambda} &= \left(\frac{\partial v^l}{\partial x^i}\frac{dx^i}{d\lambda} + v^k \Gamma^l{}_{ki} \frac{dx^i}{d\lambda} \right) e_l. \\ &= \left(\frac{\partial v^l}{\partial x^i} + v^k \Gamma^l{}_{ki} \right) \frac{dx^i}{d\lambda} e_l. \end{aligned} \tag{2.58}$$

Recalling that vectors are isomorphic to directional derivative operators, we let $u = \left(\frac{d}{d\lambda}\right)_{along\ curve}$. This is the *covariant derivative* [17, pgs. 203-208] ∇_u of v :

$$\nabla_u v = \left(\frac{\partial v^l}{\partial x^i} + \Gamma^l{}_{ki} v^k \right) \frac{dx^i}{d\lambda} e_l. \tag{2.59}$$

The coordinates of $\nabla_u v$ are

$$\nabla_i v^j = \frac{\partial v^j}{\partial x^i} + v^k \Gamma^j{}_{ik}. \tag{2.60}$$

The covariant derivative of a linear form $\omega = \omega_i \beta^i$, besides the location of indices, is the same as the covariant derivative of a vector v, except for a minus sign in front of the connection coefficients. The coordinates of a covariant derivative of a linear form are

$$\nabla_\delta \omega_\alpha = \frac{\partial \omega_\alpha}{\partial u^\delta} - \omega_\epsilon \Gamma^\epsilon{}_{\delta\alpha}. \tag{2.61}$$

Alternative notation used to denote partial and covariant derivatives is

$$\begin{aligned} \nabla_\delta v^\alpha &= v^\alpha{}_{;\delta} \\ \nabla_\delta \omega_\alpha &= \omega_{\alpha;\delta} \end{aligned} \tag{2.62}$$

and

$$\begin{aligned} \frac{\partial v^\alpha}{\partial u^\delta} &= v^\alpha{}_{,\delta} \\ \frac{\partial \omega_\alpha}{\partial u^\delta} &= \omega_{\alpha,\delta}. \end{aligned} \tag{2.63}$$

The connection coefficients are determined by requiring

$$\Gamma^j{}_{ik} = \Gamma^j{}_{ki} \tag{2.64}$$

and

$$\nabla_k g_{ij} = 0, \tag{2.65}$$

where g_{ij} is a spacetime metric. These conditions yield

$$\Gamma^j{}_{ik} = \frac{1}{2} g^{lj} \left(\frac{\partial g_{lk}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^l} \right). \tag{2.66}$$

If one wants to compare a vector p with a vector v located elsewhere, the vector p must slide over to v in such a manner that neither the length nor magnitude of p changes. This is *parallel transport* [15, section 8.5]. In terms of derivatives, parallel transport means for a curve f of parameter λ that describes the path that p followed,

$$\frac{dp}{d\lambda} = \nabla_u p = 0. \tag{2.67}$$

If the basis vectors do not shift during parallel transport (Cartesian basis vectors in Euclidean space), this expression reduces to the simple form $\nabla_u p = \frac{\partial p^l}{\partial x^i}\frac{dx^i}{d\lambda}e_l = \frac{dp^l}{d\lambda}e_l = 0$.

Geodesic equation In grade school, we are taught that the shortest distance between two points is a straight line. However, this is not the complete answer. The shortest distance between two points is a *geodesic* [15, pg. 211]. In flat space, geodesics are straight lines. On the surface of a sphere, geodesics are the great circles. An equation for geodesics can be found by *extremizing* the integral (taking the variational derivative and setting it equal to zero).

$$s = \int_P^Q \sqrt{ds^2} = \int_P^Q \sqrt{g_{ij}dx^i dx^j} = \int_P^Q \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}d\lambda. \tag{2.68}$$

It is the distance between two fixed points P and Q, where the coordinates x^i are parameterized by λ. The extremizing process yields the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0. \tag{2.69}$$

The geodesic equation can be derived in an easier way, if the concept of parallel transport is used. Consider a vector $u = u^i e_i$, which is tangent to a curve f of parameter λ. Parallel transporting u along itself yields

$$\nabla_u u = \left(\frac{\partial u^l}{\partial x^i} \frac{dx^i}{d\lambda} + \Gamma^l{}_{ki} u^k \frac{dx^i}{d\lambda} \right) e_l = 0.$$

But $\frac{\partial u^l}{\partial x^i} \frac{dx^i}{d\lambda} = \frac{du^i}{d\lambda}$ and $\frac{dx^i}{d\lambda} = u^i$, which means

$$\begin{aligned} \nabla_u u &= \left(\frac{\partial u^l}{\partial x^i} \frac{dx^i}{d\lambda} + \Gamma^l{}_{ki} u^k \frac{dx^i}{d\lambda} \right) e_l = \left(\frac{du^i}{d\lambda} + \Gamma^l{}_{ki} u^k u^i \right) e_l \qquad (2.70) \\ &= \left(\frac{d}{d\lambda} \left(\frac{dx^l}{d\lambda} \right) + \Gamma^l{}_{ki} \frac{dx^k}{d\lambda} \frac{dx^i}{d\lambda} \right) e_l = \left(\frac{d^2 x^l}{d\lambda^2} + \Gamma^l{}_{ki} \frac{dx^k}{d\lambda} \frac{dx^i}{d\lambda} \right) e_l = 0. \end{aligned}$$

This implies that $\frac{d^2 x^l}{d\lambda^2} + \Gamma^l{}_{ki} \frac{dx^k}{d\lambda} \frac{dx^i}{d\lambda} = 0$. Therefore, geodesics can be thought of in two ways. First, they are the paths that extremize the distance integral s and second, they are the paths that parallel transport their tangent vectors.

Curvature Tensors Consider the double covariant differentiation of the vector components $v^\alpha : \nabla_\nu \nabla_\mu v^\alpha$. Reversing the order of differentiation and taking the difference, yields

$$\nabla_\nu \nabla_\mu v^\alpha - \nabla_\mu \nabla_\nu v^\alpha = (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) v^\alpha. \qquad (2.71)$$

If the space is flat, the above equation is zero. If the space is curved, a nonzero result is obtained. This equation gives a coordinate-invariant measurement of curvature known as the *Riemann curvature tensor* [15, pg. 224]:

$$\begin{aligned} R^\alpha{}_{\delta\mu\nu} v^\delta &= (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) v^\alpha \Rightarrow \qquad (2.72) \\ R^\alpha{}_{\delta\mu\nu} &= (\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu) v^\alpha \omega_\delta. \end{aligned}$$

Thus, the Riemann curvature tensor is a measure of the failure of covariant derivatives to commute. The components of the Riemann curvature tensor are

$$R^{\alpha}{}_{\delta\mu\nu} = \frac{\partial \Gamma^{\alpha}{}_{\delta\upsilon}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\alpha}{}_{\delta\mu}}{\partial x^{\upsilon}} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\delta\upsilon} - \Gamma^{\alpha}{}_{\rho\upsilon}\Gamma^{\rho}{}_{\delta\mu}. \tag{2.73}$$

The Riemann curvature tensor has the following symmetries:

$$R^{\alpha}{}_{\delta\mu\nu} + R^{\alpha}{}_{\mu\nu\delta} + R^{\alpha}{}_{\nu\delta\mu} = 0 \tag{2.74}$$

$$R_{\alpha\delta\mu\nu} = -R_{\delta\alpha\mu\nu} = -R_{\alpha\delta\nu\mu} = R_{\nu\mu\alpha\delta.}$$

The Riemann curvature tensor obeys the *Bianchi identity* [15, eq. (8.46), and chp. 15]

$$R^{\alpha}{}_{\delta\mu\nu;\lambda} + R^{\alpha}{}_{\delta\lambda\mu;\nu} + R^{\alpha}{}_{\delta\nu\lambda;\mu} = 0. \tag{2.75}$$

The *Ricci curvature tensor* [15, pg. 224] is obtained by contracting the first and third indices of the Riemann curvature tensor:

$$R_{\delta\upsilon} = R^{\epsilon}{}_{\delta\epsilon\nu.} \tag{2.76}$$

The Ricci curvature tensor is symmetric and its trace is the Ricci curvature scalar:

$$R = g^{\delta\lambda}R_{\delta\lambda} = R^{\lambda}{}_{\lambda}. \tag{2.77}$$

2.2 Einstein Field Equations

Suppose an elevator containing two free particles of mass m is freely falling in a gravitational field of Newtonian gravitational potential ϕ. A simple analysis, using Newton's second law, shows that the particles don't experience a normal force from the floor of the elevator and float freely. Allowing particles to fall freely, transforms away the gravitational field. However, as hard as one could scheme, the gravitational field cannot be completely transformed away. In the elevator case above, the gravitational field is present as a tidal force. If the free particles have a separation vector $X = X^i e_i$, the components of the tidal force are given by [18, eq. (21.5)]

$$\frac{d^2 X^i}{dt^2} = -\delta^{ij} \left(\frac{\partial^2 \phi}{\partial x^j \partial x^k} \right) X^k. \tag{2.78}$$

Now, suppose the particles are freely falling in a spacetime. Because the particles are freely falling, they follow paths that are geodesics. An observer riding with one of the particles sees the neighboring particle either staying at a constant distance, moving towards, or away from the observer. If $X = X^\alpha e_\alpha$ is the separation four-vector between the particles and u is tangent to one of the geodesics, the quantitative measure of this movement is [18, eq. (21.19)]

$$\nabla_u \nabla_u X^\alpha = -R^\alpha{}_{\beta\epsilon\gamma} u^\beta X^\epsilon u^\gamma = -R^\alpha{}_{\beta\epsilon\gamma} u^\beta u^\gamma X^\epsilon. \tag{2.79}$$

This is the *geodesic deviation equation*. Comparing this equation with the tidal force equation we see that $R^\alpha{}_{\beta\epsilon\gamma} u^\beta u^\gamma$ plays a role similar to $\frac{\partial^2 \phi}{\partial x^j \partial x^k}$. Hence, gravity can be thought of as an attribute of spacetime curvature. Making an analogy with Newtonian gravity and the Bianchi identity for the Riemann curvature tensor, one is led to the Einstein vacuum field equations,

$$R_{\mu\nu} = 0, \tag{2.80}$$

and the full Einstein field equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{2.81}$$

where $T_{\mu\nu}$ is the stress-energy tensor [15, pgs. 131-132].

3 Linearized Gravity

The full Einstein field equations are a highly nonlinear system of ten second order partial differential equations. They become more tractable if they are linearized. The process of *linearization* [19, section 1.14] involves constructing the connection coefficients, Riemann curvature tensor, Ricci curvature tensor, Ricci scalar, and Einstein tensor from a metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.1}$$

with the conditions $\eta_{\mu\nu} = diag(-1,\, 1,\, 1,\, 1)$, $|h_{\mu\nu}| << 1$ and $\left|\frac{\partial h_{\mu\nu}}{\partial x^{\beta}}\right| << 1$. Due to how small $h_{\mu\nu}$ and $\frac{\partial h_{\mu\nu}}{\partial x^{\beta}}$ are, all of their products are discarded, meaning we work to first order in $h_{\mu\nu}$. Therefore, the inverse metric of $g_{\mu\nu}$ is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{3.2}$$

From the linearized equations, we obtain gravitational waves propagating through spacetime. In the following section, we derive the gravitational wave equation for linearized gravity, explore some of the properties of free gravitational waves, examine the quadrupole nature of gravitational radiation, and briefly discuss gravitational wave detection.

3.1 Linearized Field equations

In a coordinate basis, the connection coefficients, Riemann tensor and Ricci tensor components, are given by

$$\Gamma^{\delta}{}_{\beta\gamma} = g^{\alpha\delta}\frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

$$R^{\alpha}{}_{\delta\mu\nu} = \frac{\partial \Gamma^{\alpha}{}_{\delta\upsilon}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\alpha}{}_{\delta\mu}}{\partial x^{\upsilon}} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\delta\upsilon} - \Gamma^{\alpha}{}_{\rho\upsilon}\Gamma^{\rho}{}_{\delta\mu}$$

$$R^{\alpha}{}_{\delta\alpha\nu} = \frac{\partial \Gamma^{\alpha}{}_{\delta\upsilon}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\alpha}{}_{\delta\alpha}}{\partial x^{\upsilon}} + \Gamma^{\alpha}{}_{\rho\alpha}\Gamma^{\rho}{}_{\delta\upsilon} - \Gamma^{\alpha}{}_{\rho\upsilon}\Gamma^{\rho}{}_{\delta\alpha}.$$

Inserting the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ into the connection coefficients $\Gamma^{\delta}{}_{\beta\gamma}$ yields

$$\Gamma^{\delta}{}_{\beta\gamma} = \left(\eta^{\alpha\delta} - h^{\alpha\delta}\right)\frac{1}{2}\left(\frac{\partial h_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\alpha}}\right). \tag{3.3}$$

Hence, the linearized connection coefficients are

$$\delta\Gamma^{\delta}{}_{\beta\gamma} = \eta^{\alpha\delta}\frac{1}{2}\left(\frac{\partial h_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial h_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial h_{\beta\gamma}}{\partial x^{\alpha}}\right). \tag{3.4}$$

It is easy to see that the linearized Riemann and Ricci tensor coefficients are

$$\begin{aligned}
\delta R^{\alpha}{}_{\delta\mu\nu} &= \frac{\partial \Gamma^{\alpha}{}_{\delta\upsilon}}{\partial x^{\mu}} - \frac{\partial \Gamma^{\alpha}{}_{\delta\mu}}{\partial x^{\upsilon}} = \frac{1}{2}\eta^{\epsilon\alpha}\left(\frac{\partial^2 h_{\epsilon\delta}}{\partial x^{\nu}x^{\mu}} + \frac{\partial^2 h_{\epsilon\nu}}{\partial x^{\delta}x^{\mu}} - \frac{\partial^2 h_{\delta\nu}}{\partial x^{\epsilon}x^{\mu}}\right) \\
&\quad -\frac{1}{2}\eta^{\lambda\alpha}\left(\frac{\partial^2 h_{\lambda\delta}}{\partial x^{\nu}x^{\mu}} + \frac{\partial^2 h_{\lambda\mu}}{\partial x^{\delta}x^{\nu}} - \frac{\partial^2 h_{\delta\mu}}{\partial x^{\lambda}x^{\nu}}\right) \\
&= \frac{1}{2}\eta^{\omega\alpha}\left(\frac{\partial^2 h_{\omega\nu}}{\partial x^{\delta}x^{\mu}} - \frac{\partial^2 h_{\delta\nu}}{\partial x^{\omega}x^{\mu}} + \frac{\partial^2 h_{\delta\mu}}{\partial x^{\omega}x^{\nu}} - \frac{\partial^2 h_{\omega\mu}}{\partial x^{\delta}x^{\nu}}\right)
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\delta R^{\alpha}{}_{\delta\alpha\nu} &= \delta R_{\delta\nu} = \frac{1}{2}\eta^{\omega\alpha}\left(\frac{\partial^2 h_{\omega\nu}}{\partial x^{\delta}x^{\alpha}} - \frac{\partial^2 h_{\delta\nu}}{\partial x^{\omega}x^{\alpha}} + \frac{\partial^2 h_{\delta\alpha}}{\partial x^{\omega}x^{\nu}} - \frac{\partial^2 h_{\omega\alpha}}{\partial x^{\delta}x^{\nu}}\right), \\
&\quad \frac{1}{2}\left(\frac{\partial^2 h^{\alpha}{}_{\nu}}{\partial x^{\delta}x^{\alpha}} - \Box h_{\delta\nu} + \frac{\partial^2 h_{\delta}{}^{\omega}}{\partial x^{\omega}x^{\nu}} - \frac{\partial^2 h}{\partial x^{\delta}x^{\nu}}\right),
\end{aligned} \tag{3.6}$$

where $\Box = \eta^{\omega\alpha}\frac{\partial^2}{\partial x^\omega x^\alpha} = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The linearized Ricci curvature scalar is

$$\begin{aligned}
\eta^{\delta\nu}\delta R_{\delta\nu} &= \delta R = \frac{1}{2}\eta^{\delta\nu}\left(\frac{\partial^2 h^\alpha{}_\nu}{\partial x^\delta x^\alpha} - \Box h_{\delta\nu} + \frac{\partial^2 h_\delta{}^\omega}{\partial x^\omega x^\nu} - \frac{\partial^2 h}{\partial x^\delta x^\nu}\right) \\
&= \frac{1}{2}\left(\frac{\partial^2 h^{\alpha\delta}}{\partial x^\delta x^\alpha} - \Box h + \frac{\partial^2 h^{\nu\omega}}{\partial x^\omega x^\nu} - \Box h\right) \\
&= \frac{\partial^2 h^{\alpha\delta}}{\partial x^\delta x^\alpha} - \Box h,
\end{aligned} \tag{3.7}$$

where $h = Tr(h_{\alpha\beta}) = \eta^{\alpha\beta}h_{\alpha\beta}$. Using these linearized curvature quantities, one finds that the Einstein field equations, $2G_{\delta\nu} = 2R_{\delta\nu} - \eta_{\delta\nu}R = 16\pi T_{\delta\nu}$, are

$$\begin{aligned}
&\frac{\partial^2 h^\alpha{}_\nu}{\partial x^\delta x^\alpha} - \Box h_{\delta\nu} + \frac{\partial^2 h_\delta{}^\omega}{\partial x^\omega x^\nu} - \frac{\partial^2 h}{\partial x^\delta x^\nu} \\
&-\eta_{\delta\nu}\left(\frac{\partial^2 h^{\alpha\epsilon}}{\partial x^\epsilon x^\alpha} - \Box h\right) = 16\pi T_{\delta\nu}.
\end{aligned} \tag{3.8}$$

The number of terms present can be decreased by writing the above equations in terms of the trace-reverse of $h_{\delta\nu}$:

$$h'_{\delta\nu} = h_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h. \tag{3.9}$$

The perturbations $h_{\delta\nu,}$ in terms of its trace-reverse $h'_{\delta\nu}$, is $h_{\delta\nu} = h'_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h'$. Substituting this into the Einstein equations yields

$$\begin{aligned}
&\left(\frac{\partial^2 h'^\alpha{}_\nu}{\partial x^\delta x^\alpha} - \frac{1}{2}\frac{\partial^2 h'}{\partial x^\delta x^\nu}\right) - \left(\Box h'_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}\Box h'\right) + \\
&\left(\frac{\partial^2 h'^{\ \omega}_\delta}{\partial x^\omega x^\nu} - \frac{1}{2}\frac{\partial^2 h'}{\partial x^\delta x^\nu}\right) + \frac{\partial^2 h'}{\partial x^\delta x^\nu} \\
&-\eta_{\delta\nu}\left(\left(\frac{\partial^2 h'^{\alpha\epsilon}}{\partial x^\epsilon x^\alpha} - \frac{1}{2}\Box h'\right) + \Box h'\right) = 16\pi T_{\delta\nu}.
\end{aligned} \tag{3.10}$$

Grouping like terms yields the linearized field equations

$$-\Box h'_{\delta\nu} + \frac{\partial^2 h'^{\alpha}{}_{\nu}}{\partial x^\delta x^\alpha} + \frac{\partial^2 h'^{\ \omega}_{\delta}}{\partial x^\omega x^\nu} - \eta_{\delta\nu}\frac{\partial^2 h'^{\alpha\epsilon}}{\partial x^\epsilon x^\alpha} = 16\pi T_{\delta\nu}. \tag{3.11}$$

The $\Box$ operator is *the flat-space d'Alembertian.* The other terms keep the field equations gauge invariant. By requiring the perturbations $h'_{\alpha\beta}$ to satisfy the *Lorenz gauge condition,* [15, pg. 438]

$$\frac{\partial h'^{\alpha\beta}}{\partial x^\beta} = 0, \tag{3.12}$$

the terms $\frac{\partial^2 h'^{\alpha}{}_{\nu}}{\partial x^\delta x^\alpha}$, $\frac{\partial^2 h'^{\ \omega}_{\delta}}{\partial x^\omega x^\nu}$, and $\eta_{\delta\nu}\frac{\partial^2 h'^{\alpha\epsilon}}{\partial x^\epsilon x^\alpha}$ vanish and the linearized field equations reduce to

$$-\Box h'_{\delta\nu} = 16\pi T_{\delta\nu}. \tag{3.13}$$

The wave equation $-\Box h'_{\delta\nu} = 16\pi T_{\delta\nu}$, represent gravitational waves propagating through spacetime at the speed of light. Because the curvature tensors Riemann, Ricci, and Einstein, are constructed from the perturbations $h_{\delta\nu}$, gravitational waves can be thought of as curvatures propagating through spacetime. The metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the Lorenz gauge condition, and the above wave equation are the fundamental equations of linearized gravity written in the Lorenz gauge. Before moving on, it is important to see how perturbations $h'_{\alpha\beta}$ are transformed into others that satisfy the Lorenz gauge condition. Consider how $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ transforms under an arbitrary infinitesimal coordinate transformation

$$x''^\mu = x^\mu + \xi^\mu. \tag{3.14}$$

Transforming the metric tensor into these new coordinates yields

$$
\begin{aligned}
g''_{\alpha\beta} &= g_{\mu\nu}\frac{\partial x^\mu}{\partial x''^\alpha}\frac{\partial x^\nu}{\partial x''^\beta} \\
&= (\eta_{\mu\nu}+h_{\mu\nu})\left(\delta^\mu_\alpha-\frac{\partial \xi^\mu}{\partial x''^\alpha}\right)\left(\delta^\nu_\beta-\frac{\partial \xi^\nu}{\partial x''^\beta}\right) \\
&= (\eta_{\mu\nu}+h_{\mu\nu})\left(\delta^\mu_\alpha\delta^\nu_\beta-\delta^\mu_\alpha\frac{\partial \xi^\nu}{\partial x''^\beta}-\delta^\nu_\beta\frac{\partial \xi^\mu}{\partial x''^\alpha}+\frac{\partial \xi^\mu}{\partial x''^\alpha}\frac{\partial \xi^\nu}{\partial x''^\beta}\right) \\
&= \eta_{\alpha\beta}+h_{\alpha\beta}-\frac{\partial \xi_\alpha}{\partial x''^\beta}-\frac{\partial \xi_\beta}{\partial x''^\alpha}+\frac{\partial \xi_\nu}{\partial x''^\alpha}\frac{\partial \xi^\nu}{\partial x''^\beta} \\
&\quad -h_{\alpha\nu}\frac{\partial \xi^\nu}{\partial x''^\beta}-h_{\mu\beta}\frac{\partial \xi^\mu}{\partial x''^\alpha}+h_{\mu\nu}\frac{\partial \xi^\mu}{\partial x''^\alpha}\frac{\partial \xi^\nu}{\partial x''^\beta}.
\end{aligned}
\tag{3.15}
$$

Since ξ^μ and its derivatives are infinitesimal in size and $|h_{\alpha\beta}| << 1$, all products of $h_{\alpha\nu}$ with derivatives of ξ^μ are dropped. The term $\frac{\partial \xi_\nu}{\partial x''^\alpha}\frac{\partial \xi^\nu}{\partial x''^\beta}$ is also dropped. To first order in $h_{\alpha\beta}$ and $\frac{\partial \xi_\nu}{\partial x''^\beta}$, the metric tensor is

$$
g''_{\alpha\beta} = \eta_{\alpha\beta}+h_{\alpha\beta}-\frac{\partial \xi_\alpha}{\partial x''^\beta}-\frac{\partial \xi_\beta}{\partial x''^\alpha} = \eta_{\alpha\beta}+h''_{\alpha\beta}. \tag{3.16}
$$

Thus, under an infinitesimal coordinate transformation the perturbations $h_{\alpha\beta}$ transform as

$$
h''_{\alpha\beta} = h_{\alpha\beta}-\frac{\partial \xi_\alpha}{\partial x''^\beta}-\frac{\partial \xi_\beta}{\partial x''^\alpha}. \tag{3.17}
$$

Now, the linearized Einstein equations are written in terms of the trace-reverse of $h_{\alpha\beta}$, which is $h'_{\alpha\beta} = h_{\alpha\beta}-\frac{1}{2}\eta_{\alpha\beta}h$. Under an infinitesimal coordinate transformation, the trace-reverse of $h_{\alpha\beta}$ transforms as

$$\begin{aligned}
\overline{h}'_{\alpha\beta} &= \left(h_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha}\right) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\epsilon\mu}h''_{\epsilon\mu} \\
&= \left(h_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha}\right) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\epsilon\mu}\left(h_{\epsilon\mu} - \frac{\partial\xi_\epsilon}{\partial x''^\mu} - \frac{\partial\xi_\mu}{\partial x''^\epsilon}\right) \\
&= \left(h_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha}\right) - \frac{1}{2}\eta_{\alpha\beta}\left(h - \frac{\partial\xi^\mu}{\partial x''^\mu} - \frac{\partial\xi^\epsilon}{\partial x''^\epsilon}\right) \\
&= \left(h_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha}\right) - \frac{1}{2}\eta_{\alpha\beta}\left(h - 2\frac{\partial\xi^\gamma}{\partial x''^\gamma}\right) \\
&= \left(h_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha}\right) - \frac{1}{2}\eta_{\alpha\beta}h + \eta_{\alpha\beta}\frac{\partial\xi^\gamma}{\partial x''^\gamma} \\
&= h'_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x''^\beta} - \frac{\partial\xi_\beta}{\partial x''^\alpha} + \eta_{\alpha\beta}\frac{\partial\xi^\gamma}{\partial x''^\gamma}.
\end{aligned} \tag{3.18}$$

Raising the indices of $\overline{h}'_{\alpha\beta}$ by the inverse Minkowski metrics $\eta^{\alpha\lambda}$ and $\eta^{\epsilon\beta}$ gives

$$\overline{h}'^{\lambda\epsilon} = h'^{\lambda\epsilon} - \eta^{\epsilon\beta}\frac{\partial\xi^\lambda}{\partial x''^\beta} - \eta^{\lambda\alpha}\frac{\partial\xi^\epsilon}{\partial x''^\alpha} + \eta^{\lambda\epsilon}\frac{\partial\xi^\gamma}{\partial x''^\gamma}. \tag{3.19}$$

Applying the Lorenz gauge condition to $\overline{h}'^{\lambda\epsilon}$ yields

$$\frac{\partial\overline{h}'^{\lambda\epsilon}}{\partial x''^\epsilon} = \frac{\partial h'^{\lambda\epsilon}}{\partial x''^\epsilon} - \eta^{\epsilon\beta}\frac{\partial^2\xi^\lambda}{\partial x''^\beta\partial x''^\epsilon} - \eta^{\lambda\alpha}\frac{\partial^2\xi^\epsilon}{\partial x''^\alpha\partial x''^\epsilon} + \eta^{\lambda\epsilon}\frac{\partial^2\xi^\gamma}{\partial x''^\gamma\partial x''^\epsilon} = 0. \tag{3.20}$$

The terms $\eta^{\lambda\alpha}\frac{\partial^2\xi^\epsilon}{\partial x''^\alpha\partial x''^\epsilon}$ and $\eta^{\lambda\alpha}\frac{\partial^2\xi^\epsilon}{\partial x''^\alpha\partial x''^\epsilon}$ are identical, meaning that the above expression reduces to

$$\frac{\partial\overline{h}'^{\lambda\epsilon}}{\partial x''^\epsilon} = \frac{\partial h'^{\lambda\epsilon}}{\partial x''^\epsilon} - \eta^{\epsilon\beta}\frac{\partial^2\xi^\lambda}{\partial x''^\beta\partial x''^\epsilon} = 0. \tag{3.21}$$

The operator $\eta^{\epsilon\beta}\frac{\partial^2}{\partial x''^\beta\partial x''^\epsilon}$ is the flat-space d'Alembertian, $\Box$, meaning that the above equation may be written as

$$\Box\xi^\lambda = \frac{\partial h'^{\lambda\epsilon}}{\partial x''^\epsilon}. \tag{3.22}$$

Thus, the Lorenz gauge condition, $\frac{\partial \overline{h}'^{\lambda\epsilon}}{\partial x''^{\epsilon}} = 0$, imposes the flat-space wave equation as a constraint on the infinitesimal coordinate translations, ξ^{λ}. If the perturbations $h'^{\lambda\epsilon}$ do not satisfy the Lorenz gauge condition, then performing an infinitesimal coordinate transformation of the form $x''^{\mu} = x^{\mu} + \xi^{\mu}$, where ξ^{μ} satisfies the wave equation $\Box\xi^{\lambda} = \frac{\partial h'^{\lambda\epsilon}}{\partial x''^{\epsilon}}$, guarantees that the resulting perturbation $\overline{h}'^{\lambda\epsilon}$ satisfies the Lorenz gauge condition. If $h'^{\lambda\epsilon}$ satisfies the Lorenz gauge condition, then the transformation $x''^{\mu} = x^{\mu} + \xi^{\mu}$ takes $h'^{\lambda\epsilon}$ into other perturbations $\overline{h}'^{\lambda\epsilon}$ that also satisfy the Lorenz gauge condition, if $\Box\xi^{\lambda} = 0$.

3.2 Free Gravitational Waves and TT-Gauge

3.2.1 Free Gravitational Waves

Free gravitational waves at a sufficient large distance from their source are solutions to the homogenous wave-equation

$$\Box h'_{\delta\nu} = 0. \tag{3.23}$$

The most simple solution to this equation is the gravitational plane wave [19, eq. (1.14.21)]

$$h'_{\delta\nu} = A'_{\delta\nu} e^{ik_{\beta}x^{\beta}}. \tag{3.24}$$

Inserting this into the homogenous wave equation yields

$$\begin{aligned}
\Box h'_{\delta\nu} &= \Box\left(A'_{\delta\nu}e^{ik_\beta x^\beta}\right) = \eta^{\alpha\epsilon}\frac{\partial^2}{\partial x^\alpha \partial x^\epsilon}\left(A'_{\delta\nu}e^{ik_\beta x^\beta}\right) \\
&= -A'_{\delta\nu}e^{ik_\beta x^\beta}\eta^{\alpha\epsilon}\left(k_\beta\delta^\beta_\alpha k_\beta\delta^\beta_\epsilon\right) \\
&= -A'_{\delta\nu}e^{ik_\beta x^\beta}\left(\eta^{\alpha\epsilon}k_\alpha k_\epsilon\right) = -A'_{\delta\nu}e^{ik_\beta x^\beta}\left(k_\alpha k^\alpha\right) = 0.
\end{aligned} \tag{3.25}$$

This implies that the scalar product $k_\alpha k^\alpha$ equals zero, meaning that the wave vector k^β is light-like. The $h'_{\delta\nu}$ also satisfy the Lorenz gauge condition $\frac{\partial h'^{\delta\nu}}{\partial x^\nu} = 0$:

$$\frac{\partial h'^{\delta\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(A'^{\delta\nu}e^{ik_\epsilon x^\epsilon}\right) = i\left(A'^{\delta\nu}k_\nu\right) = 0. \tag{3.26}$$

This result means that the amplitude tensor $A'_{\delta\nu}$ is orthogonal to the wave vector k^ν. If we envision a gravitational plane wave passing through a region of flat spacetime, the only place in this region that has nonzero curvature is where the gravitational wave passes through. To first order in $h_{\delta\nu}$ in terms of the trace-reversed perturbations $h'_{\delta\nu}$, the Riemann curvature tensor of this location is

$$\begin{aligned}
\delta R^\alpha{}_{\delta\mu\nu} &= \frac{1}{2}\eta^{\omega\alpha}\left(\frac{\partial^2 h_{\omega\nu}}{\partial x^\delta x^\mu} - \frac{\partial^2 h_{\delta\nu}}{\partial x^\omega x^\mu} + \frac{\partial^2 h_{\delta\mu}}{\partial x^\omega x^\nu} - \frac{\partial^2 h_{\omega\mu}}{\partial x^\delta x^\nu}\right) \\
&= \frac{1}{2}\eta^{\omega\alpha}(\frac{\partial^2}{\partial x^\delta x^\mu}\left(h'_{\omega\nu} - \frac{1}{2}\eta_{\omega\nu}h'\right) - \frac{\partial^2}{\partial x^\omega x^\mu}\left(h'_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h'\right) \\
&\quad + \frac{\partial^2}{\partial x^\omega x^\nu}\left(h'_{\delta\mu} - \frac{1}{2}\eta_{\delta\mu}h'\right) - \frac{\partial^2}{\partial x^\delta x^\nu}\left(h'_{\omega\mu} - \frac{1}{2}\eta_{\omega\mu}h'\right)) \\
&= \frac{1}{2}\eta^{\omega\alpha}(\left(\frac{\partial^2 h'_{\omega\nu}}{\partial x^\delta x^\mu} - \frac{\partial^2 h'_{\delta\nu}}{\partial x^\omega x^\mu} + \frac{\partial^2 h'_{\delta\mu}}{\partial x^\omega x^\nu} - \frac{\partial^2 h'_{\omega\mu}}{\partial x^\delta x^\nu}\right) \\
&\quad + \frac{1}{2}\left(-\eta_{\omega\nu}\frac{\partial^2}{\partial x^\delta x^\mu} + \eta_{\delta\nu}\frac{\partial^2}{\partial x^\omega x^\mu} - \eta_{\delta\mu}\frac{\partial^2}{\partial x^\omega x^\nu} + \eta_{\omega\mu}\frac{\partial^2}{\partial x^\delta x^\nu}\right)h') \\
&= \frac{1}{2}\eta^{\omega\alpha}((-A'_{\omega\nu}k_\delta k_\mu + A'_{\delta\nu}k_\omega k_\mu - A'_{\delta\mu}k_\omega k_\nu + A'_{\omega\mu}k_\delta k_\nu) \\
&\quad + \frac{1}{2}\left(-\eta_{\omega\nu}k_\delta k_\mu + \eta_{\delta\nu}k_\omega k_\mu - \eta_{\delta\mu}k_\omega k_\nu + \eta_{\omega\mu}k_\delta k_\nu\right)A')e^{ik_\beta x^\beta}
\end{aligned} \tag{3.27}$$

Consider the degrees of freedom of $h'_{\delta\nu}$. Symmetrically, since $h'_{\delta\nu}$ is part of the metric $g_{\delta\nu} = \eta_{\delta\nu} + h_{\delta\nu} = \eta_{\delta\nu} + h'_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h'$, the degrees of freedom of $h'_{\delta\nu}$ cannot be more than ten. However, of these ten degrees of freedom, four are fixed by the gauge invariance of $g_{\delta\nu}$ and four others are fixed by the Lorenz gauge condition $\frac{\partial h'^{\delta\nu}}{\partial x^{\nu}} = 0$. Of ten possible degrees of freedom, only two remain. Eight of the ten degrees of freedom are fixed by gauge invariance and the Lorenz gauge condition. This implies that gravitational fields generated by astrophysical sources have only two degrees of freedom. Because at sufficiently large distances, gravitational fields of astrophysical sources can be described by metrics of the form $g_{\delta\nu} = \eta_{\delta\nu} + h_{\delta\nu} = \eta_{\delta\nu} + h'_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h'$ where $|h'_{\delta\nu}| << 1$.

3.2.2 Traverse Traceless Gauge (TT-Gauge)

For free gravitational waves, it is convenient to use an infinitesimal coordinate transformation $x''^{\mu} = x^{\mu} + \xi^{\mu}$ with the restriction $\Box\xi^{\mu} = 0$ to transform the perturbations $h'_{\delta\nu}$ into others that satisfy the flat-space sourceless wave equation, the Lorenz gauge condition, and have the properties [15, secton 35.4]

$$\overline{h}'_{\delta 0} = 0 \tag{3.28(a)}$$

$$\overline{h}'^{\nu}{}_{\nu} = 0. \tag{3.28(b)}$$

If this is done, we have chosen a particular gauge in which to work, namely the *traverse-traceless gauge (TT-gauge)*. Now, we will explicitly work out the transformation of $h'_{\delta\nu}$ to the traverse-traceless perturbations $\overline{h}'_{\delta\nu}$. To transform $h'_{\delta\nu}$ to the TT-gauge, we use the transformation $\overline{h}'_{\alpha\beta} = h'_{\alpha\beta} - \frac{\partial\xi_\alpha}{\partial x^\beta} - \frac{\partial\xi_\beta}{\partial x^\alpha} + \eta_{\alpha\beta}\frac{\partial\xi^\lambda}{\partial x^\lambda}$ and the *imaginary plane-wave solutions* of $\Box\xi_\mu = 0$. The imaginary plane-wave solutions of $\Box\xi_\mu = 0$ are

$$\xi_\mu = -iC_\mu e^{ik_\gamma x^\gamma}. \tag{3.29}$$

After transforming $h'_{\delta\nu}$ to $\overline{h}'_{\delta\nu}$ and letting the ξ_μ have the above form, putting $\overline{h}'_{\delta\nu}$ into the TT-gauge is equivalent to solving for the four constants C_μ. Starting with condition one, $\overline{h}'_{\delta 0} = 0$, we have

$$\overline{h}'_{\alpha 0} = h'_{\alpha 0} - \frac{\partial\xi_\alpha}{\partial x^0} - \frac{\partial\xi_0}{\partial x^\alpha} + \eta_{\alpha 0}\frac{\partial\xi^\gamma}{\partial x^\gamma} = 0. \tag{3.30}$$

The derivatives of ξ_μ are

$$\frac{\partial\xi_\alpha}{\partial x^0} = k_0 C_\alpha e^{ik_\gamma x^\gamma}, \tag{3.31}$$

$$\frac{\partial\xi_0}{\partial x^\alpha} = k_\alpha C_0 e^{ik_\gamma x^\gamma}, \tag{3.32}$$

and

$$\frac{\partial\xi^\lambda}{\partial x^\lambda} = \eta^{\lambda\epsilon} k_\epsilon C_\lambda e^{ik_\gamma x^\gamma}. \tag{3.33}$$

Thus, condition one, $\overline{h}'_{\delta 0} = 0$, yields four equations:

$$h'_{00} - (k_0 C_0 + k_1 C_1 + k_2 C_2 + k_3 C_3)\, e^{ik_\gamma x^\gamma} = 0 \tag{3.34(a)}$$

$$h'_{10} - (k_0 C_1 + k_1 C_0)\, e^{ik_\gamma x^\gamma} = 0 \tag{3.34(b)}$$

$$h'_{20} - (k_0 C_2 + k_2 C_0)\, e^{ik_\gamma x^\gamma} = 0 \tag{3.34(c)}$$

$$h'_{30} - (k_0 C_3 + k_3 C_0)\, e^{ik_\gamma x^\gamma} = 0. \tag{3.34(d)}$$

Moving to condition two, $\overline{h}'^{\nu}{}_{\nu} = 0$, we have

$$\overline{h}'^{\nu}{}_{\nu} = h'^{\nu}{}_{\nu} - \eta^{\mu\nu}\frac{\partial \xi_\mu}{\partial x^\nu} - \eta^{\mu\nu}\frac{\partial \xi_\nu}{\partial x^\mu} + \frac{\partial \xi^\lambda}{\partial x^\lambda} = 0. \tag{3.35}$$

Inserting the above derivatives of ξ_μ into $\overline{h}'^{\nu}{}_{\nu} = 0$ yields

$$\begin{aligned} & h'^{\nu}{}_{\nu} - \eta^{\mu\nu}\left(k_\nu C_\mu e^{ik_\gamma x^\gamma}\right) - \eta^{\mu\nu}\left(k_\mu C_\nu e^{ik_\gamma x^\gamma}\right) + \eta^{\lambda\epsilon} k_\epsilon C_\lambda e^{ik_\gamma x^\gamma} \\ & = h'^{\nu}{}_{\nu} - \eta^{\mu\nu}\left(k_\nu C_\mu e^{ik_\gamma x^\gamma}\right) \\ & = -h'_{00} + h'_{11} + h'_{22} + h'_{33} - \left(-k_0 C_0 + k_1 C_1 + k_2 C_2 + k_3 C_3\right) e^{ik_\gamma x^\gamma} = 0. \end{aligned} \tag{3.36}$$

Using equations (3.34(a)-(d)) and equation (3.36), the constants C_0, C_1, C_2, and C_3 can be determined. Subtracting equation (3.34(a)) from equation (3.36) yields

$$-2h'_{00} + h'_{11} + h'_{22} + h'_{33} + 2k_0 C_0 e^{ik_\gamma x^\gamma} = 0. \tag{3.37}$$

Algebraic manipulation gives

$$\begin{aligned} C_0 &= \frac{e^{-ik_\gamma x^\gamma}}{2k_0}\left(h'_{00} - h'_{11} - h'_{22} - h'_{33}\right) \\ &= \frac{e^{-ik_\gamma x^\gamma}}{2k_0}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right) e^{ik_\gamma x^\gamma} \\ &= \frac{1}{2k_0}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right). \end{aligned} \tag{3.38}$$

Next, solving for C_1 from equation (3.34(a)) leads to

$$\begin{aligned} C_1 &= \frac{e^{-ik_\gamma x^\gamma}}{k_0} h'_{10} - \frac{k_1}{k_0} C_0 \\ &= \frac{e^{-ik_\gamma x^\gamma}}{k_0} A'_{10} e^{ik_\gamma x^\gamma} - \frac{k_1}{k_0}\left(\frac{1}{2k_0}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right)\right) \\ &= \frac{1}{k_0} A'_{10} - \frac{k_1}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right). \end{aligned} \tag{3.39}$$

Likewise, equations (3.34(c)) and (3.34(d)) yield

$$C_2 = \frac{1}{k_0} A'_{20} - \frac{k_2}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right), \tag{3.40}$$

and

$$C_3 = \frac{1}{k_0} A'_{30} - \frac{k_3}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right). \tag{3.41}$$

Equations (3.38-3.41) are the required constants. Thus, the coordinate translations ξ_μ needed to put $\overline{h}'_{\delta\nu}$ into the TT-gauge are

$$\begin{aligned}
\xi_0 &= \frac{-i}{2k_0}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right) e^{ik_\gamma x^\gamma} \\
\xi_1 &= -i\left(\frac{1}{k_0} A'_{10} - \frac{k_1}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right)\right) e^{ik_\gamma x^\gamma} \\
\xi_2 &= -i\left(\frac{1}{k_0} A'_{20} - \frac{k_2}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right)\right) e^{ik_\gamma x^\gamma} \\
\xi_3 &= -i\left(\frac{1}{k_0} A'_{30} - \frac{k_3}{2\left(k_0\right)^2}\left(A'_{00} - A'_{11} - A'_{22} - A'_{33}\right)\right) e^{ik_\gamma x^\gamma}.
\end{aligned} \tag{3.42}$$

The advantage of the TT-gauge is that gravitational waves take on a simple form. Supposing that our spacetime coordinates are the tetrad $(t,\ x,\ y,\ z)$, the tensor describing a traverse-traceless gravitational wave, $h^{TT}_{\delta\nu}$, traveling in the $z-$direction is

$$h^{TT}_{\delta\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{ik_\beta x^\beta}. \tag{3.43}$$

The coefficients a_+ and $a_\times$ represent the two degrees of freedom of the wave and are called *the plus and cross polarization modes* [15, section 35.6]. The names "plus" and "cross" arise from how a gravitational wave in each of these polarization modes affects a ring of free particles. A gravitational wave can be a linear combination of both polarization modes.

For a ring of freely falling particles in the $x - y$ plane of the tetrad (t, x, y, z), a "plus" polarized gravitational wave passing through the ring of particles in the $z-$direction would cause the ring of particles to stretch in the $x-$direction and be squeezed in the $y-$direction. Then, it is stretched in the $y-$direction and squeezed in the $x-$direction. If a "cross" polarized gravitational wave passes through the ring of particles in the $z-$direction, then the ring of particles would stretch in a diagonal direction of 45° to the $y-$axis and be squeezed in a second diagonal direction of 90° to the first. Then, it would stretch in the second diagonal direction and be squeezed in the diagonal direction of 45° to the $y-$axis. We can easily calculate the tidal force experienced by two of these particles by calculating the Riemann tensor components in the TT-gauge (k_0 is the angular frequency ω of the wave) [15, pg. 948]:

$$R_{j0k0} = R_{0j0k} = -R_{j00k} = -R_{0jk0} = -\frac{1}{2}h^{TT}_{jk,00} = \frac{1}{2}\left(k_0\right)^2 A^{TT}_{jk}\, e^{ik_\beta x^\beta} = \frac{1}{2}\omega^2 A^{TT}_{jk}\, e^{ik_\beta x^\beta}. \tag{3.44}$$

Since the gravitational wave is traveling in the $z-$direction, these components are

$$R_{x0x0} = \frac{1}{2}\omega^2 a_+ e^{i(\omega t + k_z z)}, \tag{3.45(a)}$$

$$R_{y0y0} = -\frac{1}{2}\omega^2 a_+ e^{i(\omega t + k_z z)}, \tag{3.45(b)}$$

$$R_{x0y0} = \frac{1}{2}\omega^2 a_\times e^{i(\omega t + k_z z)}, \tag{3.45(c)}$$

and

$$R_{y0x0} = \frac{1}{2}\omega^2 a_\times e^{i(\omega t + k_z z)}. \tag{3.45(c)}$$

If the two particles are separated by the vector χ^α, then the tidal force they experience from these gravitational waves is

$$\begin{aligned} \nabla_u \nabla_u X^\alpha &= -R^\alpha{}_{\beta\epsilon\gamma} u^\beta X^\epsilon u^\gamma \\ \nabla_u \nabla_u X^\alpha &= a^\alpha = -R^\alpha{}_{0\epsilon 0} X^\epsilon. \end{aligned} \tag{3.46}$$

Hence, the tidal force in the x and y directions are

$$a^x = -\eta^{x\lambda} R_{\lambda 0\epsilon 0} X^\epsilon = -\left(R_{x0x0} X^x + R_{x0y0} X^y\right) = -\frac{1}{2}\omega^2 \left(a_+ X^x + a_\times X^y\right) e^{i(\omega t + k_z z)}, \tag{3.47}$$

and

$$a^y = -\eta^{y\lambda} R_{\lambda 0\epsilon 0} X^\epsilon = -\left(R_{y0y0} X^y + R_{y0x0} X^x\right) = -\frac{1}{2}\omega^2 \left(a_\times X^x - a_+ X^y\right) e^{i(\omega t + k_z z)}. \tag{3.48}$$

Even though these particles experience a tidal force, they are not moved to new coordinates. However, the distance between them is changed by the gravitational waves. To illustrate that the gravitational waves do not move our two particles to new coordinates, label these particles as 1 and 2 and suppose particle 1 is at the origin $(0, 0, 0)$ and particle 2 is at the point $(x,\ y, 0)$. Because the particles are freely falling, both follow geodesics, meaning that the equations of motion for these particles are

$$\frac{d^2 x_1^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\epsilon\mu} \frac{dx_1^\epsilon}{d\tau} \frac{dx_1^\mu}{d\tau} \tag{3.49}$$

and

$$\frac{d^2x_2^\beta}{d\tau^2} = -\Gamma^\beta{}_{\epsilon\mu}\frac{dx_2^\epsilon}{d\tau}\frac{dx_2^\mu}{d\tau}. \tag{3.50}$$

Suppose that the initial four-velocities of the particles are $\frac{dx_1^\mu}{d\tau} = \frac{dx_2^\mu}{d\tau} = (1,0,0,0)$.

The equations of motion for particles 1 and 2 are

$$\frac{d^2x_1^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{00} \tag{3.51}$$

$$\frac{d^2x_2^\beta}{d\tau^2} = -\Gamma^\beta{}_{00}. \tag{3.52}$$

Because we are working in linearized gravity, $x_1^\alpha = \delta x_1^\alpha$, $x_2^\beta = \delta x_2^\beta$, and $\Gamma^\lambda{}_{00} = \delta\Gamma^\lambda{}_{00}$. This means that the equations of motion are

$$\frac{d^2\delta x_1^\alpha}{d\tau^2} = -\delta\Gamma^\alpha{}_{00} = -\eta^{\alpha\delta}\frac{1}{2}\left(\frac{\partial h_{\delta 0}}{\partial x^0} + \frac{\partial h_{0\delta}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\delta}\right) \tag{3.53}$$

$$\frac{d^2\delta x_2^\beta}{d\tau^2} = -\delta\Gamma^\beta{}_{00} = -\eta^{\beta\delta}\frac{1}{2}\left(\frac{\partial h_{\delta 0}}{\partial x^0} + \frac{\partial h_{0\delta}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\delta}\right). \tag{3.54}$$

Since the gravitational waves satisfy $\Box h'_{\delta\nu} = \Box\left(h_{\delta\nu} - \frac{1}{2}\eta_{\delta\nu}h\right) = 0$, choose $h'_{\delta\nu}$ to be in the TT-gauge. This forces $h_{\delta\nu}$ to also be in the TT-gauge. This yields

$$\frac{d^2\delta x_1^\alpha}{d\tau^2} = 0, \tag{3.55}$$

and

$$\frac{d^2\delta x_2^\beta}{d\tau^2} = 0. \tag{3.56}$$

Thus, there is no change in either particle's coordinates. Now, let $L = (0,\ L^x,\ L^y,\ 0)$

be the separation vector of the two particles. Calculate the distance between them:

$$\begin{aligned} d &= \sqrt{g(L,L)} = \sqrt{g_{ij}L^iL^j} \\ &= \sqrt{\eta_{ij}L^iL^j + h_{ij}L^iL^j} \\ &= L\sqrt{1 + h_{ij}\frac{L^iL^j}{L^2}}. \end{aligned} \tag{3.57}$$

Performing a binomial expansion to first order, since the h_{ij} are small, yields

$$d = L\sqrt{1 + h_{ij}\frac{L^iL^j}{L^2}} \approx L\left(1 + \frac{1}{2}h_{ij}\frac{L^iL^j}{L^2}\right) \tag{3.58}$$

Still supposing that $h'_{\delta\nu}$ is still in the TT-gauge, and that the gravitational waves are traveling in the $z-$direction, the perturbations h_{ij} have the form

$$h_{ij}(t,0) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_+ & a_\times & 0 \\ 0 & a_\times & -a_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega t}. \tag{3.59}$$

This yields

$$\begin{aligned} d &= L\left(1 + \frac{1}{2L^2}\left(h_{xx}L^xL^x + 2h_{xy}L^xL^y + h_{yy}L^yL^y\right)\right) \\ &= L\left(1 + \frac{1}{2L^2}\left(a_+\left(L^xL^x - L^yL^y\right) + 2a_\times L^xL^y\right)e^{i\omega t}\right). \end{aligned} \tag{3.60}$$

Thus, the distance between the particles oscillates as the gravitational wave passes through the $x-y$ plane. In the equation for distance between the particles, the part that gives the change in distance is

$$\delta d = \frac{1}{2}h_{ij}\frac{L^iL^j}{L}. \tag{3.61}$$

Dividing both sides by L yields

$$\frac{\delta d}{L} = \frac{1}{2} h_{ij} \frac{L^i L^j}{L^2}. \tag{3.62}$$

The the change in length per length, $\frac{\delta d}{L}$, is called a *strain*. This shows that the perturbations $h_{\alpha\beta}$ can be thought of as propagating spacetime curvature, or as a strain propagating through spacetime.

3.2.3 Quadrupole Moment

By studying the sourceless gravitational wave equation $\Box h'_{\delta\nu} = 0$, it is found that gravitational waves have only two degrees of freedom. Since any gravitational wave from a real astrophysical source will satisfy $\Box h'_{\delta\nu} = 0$ at a sufficiently large distance, it can be inferred that a gravitational field only has two dynamical degrees of freedom. Suppose we have the non-vacuum case, $-\Box h'^{\delta\nu} = 16\pi T^{\delta\nu}$. The general solution to this equation is [18, pg. 496]

$$h'^{\delta\nu}(t, \vec{x}) = 4 \int d^3x' \frac{T^{\delta\nu}(t', \vec{x}')}{\left| \vec{x} - \vec{x}' \right|}, \tag{3.63}$$

where t' is the retarded time $t' = t_{ret} \equiv t - \left| \vec{x} - \vec{x}' \right|$. To ensure that the perturbations $h'_{\delta\nu}(t, \vec{x})$ are small disturbances of Minkowski space, we will suppose that the gravitational wave source is weak. This means that the constituents of the gravitating system move at speeds much less than the speed of light. It also means that we are far enough away from the source and that $r = \left| \vec{x} - \vec{x}' \right| \approx \left| \vec{x} \right|$.

If B_{source} is the characteristic size of the source, these requirements mean that $r >> B_{source}$. For the wavelength λ of the gravitational waves, $\lambda >> B_{source}$. Requiring the source to move at speeds much less than the speed of light, implies that the stress-energy tenors $T^{\delta\nu}$ is dominated by rest-mass energy density ϵ. Hence, the stress energy tensor that models our weak source can be written as

$$T^{\delta\nu} = \epsilon u^{\delta} u^{\nu}, \tag{3.64}$$

where u^{α} is the four velocity of the source. With these requirements, the gravitational wave amplitudes $h'^{\delta\nu}$ in terms of a source $T^{\delta\nu}$are [18, pgs. 498-499].

$$h'^{\delta\nu}(t, \vec{x}) = \frac{4}{r} \int d^3x' T^{\delta\nu}(t - r, \vec{x}'). \tag{3.65}$$

In terms of the second mass moment

$$I^{ij} = \int d^3x T^{00} x^i x^j \tag{3.66}$$

the spatial components h'^{ij} of $h'^{\delta\nu}$ can be written as [15, pg. 1001]

$$h'^{ij} = \frac{2}{r} \ddot{I}^{ij}(t - r). \tag{3.67}$$

In electromagnetism, the lowest order multipole moment that radiates is the *dipole moment.* For a gravitating system however, there is no radiation from the dipole moment. To see this in our weak source, long wavelength approximation, consider the perturbations

$$h'^{t\nu} = \frac{4}{r} \int d^3x' T^{t\nu} = \frac{4}{r} \int d^3x' \epsilon u^t u^{\nu} = \frac{4}{r} \int d^3x' \epsilon u^{\nu}. \tag{3.68}$$

Because $\int d^3x'\epsilon u^\nu$ dimensionally is mass times velocity, and u^ν is the four velocity of the matter, we have the total energy-momentum four-vector

$$P^\nu = \int d^3x'\epsilon u^\nu. \tag{3.69}$$

Thus,

$$h'^{t\nu} = \frac{4}{r}P^\nu. \tag{3.70}$$

Now, consider

$$h'^{ti} = \frac{4}{r}\int d^3x'\epsilon u^i. \tag{3.71}$$

In terms of the proper time τ, the spatial part of the four-velocity is $u^i = \frac{dx'^i}{d\tau}$. Recalling that the mass dipole moment is

$$p^i = \int d^3x'\epsilon x'^i \tag{3.72}$$

and that in our approximation ϵ is constant, we see that

$$h'^{ti} = \frac{4}{r}\int d^3x'\epsilon u^i = \frac{4}{r}\frac{dp^i}{d\tau}. \tag{3.73}$$

Next, recall that the definition of a system's center of mass is

$$R^i_{CM} = \frac{\int d^3x'\epsilon x'^i}{\int d^3x'\epsilon}. \tag{3.74}$$

In terms of the center of mass the system's dipole moment is

$$p^i = R^i_{CM}\int d^3x'\epsilon. \tag{3.75}$$

Hence, the h'^{ti} may be written as

$$h'^{ti} = \frac{4}{r}\frac{dR^i_{CM}}{d\tau}\int d^3x'\epsilon. \tag{3.76}$$

However, we can transfer to an inertial frame where R^i_{CM} is at the origin. Doing this yields $R^i_{CM} = 0$, $p^i = 0$ and $h'^{ti} = 0$. The Lorentz transformations required are in the direction of the center of mass and have velocity

$$v = \sqrt{\eta_{\alpha\beta}u^\alpha u^\beta}. \tag{3.77}$$

Therefore, the mass dipole can be transformed away, meaning that there can be no mass dipole radiation. The lowest order mass multipole moment that radiates is the quadrupole moment. Hence, the power output due to gravitational radiation of a gravitating system is [18, pg. 507]

$$L_{mass\ quadrupole} = \frac{1}{5}\left\langle \dddot{\breve{I}}_{ij}\dddot{\breve{I}}^{ij}\right\rangle, \tag{3.78}$$

where

$$\breve{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I. \tag{3.79}$$

3.3 Detecting Gravitational Waves

Detecting a gravitational wave amounts to calculating the strain induced by the wave [20, 21]. In recent times (2002-2010) the main type of instruments used to detect these strains are laser interferometers. In their basic form, they are *Micholson-Morely interferometers*, with the light source being a laser. How these instruments detect gravitational waves is simple. Two test masses with completely reflecting mirrors attached, are at the end of each interferometer arm. At the junction where the arms intersect is a beam splitter. The light from the laser is split and half of the beam travels through each arm. After the split beam is reflected at each mirror, the beam recombines at the beam splitter and goes to the photodetector. It's at the photodetector that the beam's interference pattern as a function of time is recorded.

If no gravitational waves pass through the interferometer, the photodetector records no light. However, if a gravitational wave passes through the interferometer, the laser beam in each arm of the interferometer will travel different distances. This causes the photodetector to record a time varying interference pattern. Contained within this interference pattern are the gravitational waveforms. This is the basic description of laser interferometers. To turn this basic design into a real laser interferometer, several features would need to be added. The first feature is to add a second test mass with a partially reflecting mirror in each arm. This turns each arm into a *Fabry-Perot cavity.* Another major feature would be the addition of a mirror between the laser and the beam splitter. This recycles any light that makes it past the beam splitter after it exits an arm. Other features that would need to be added to the interferometer are vibration isolators and seismic attenuators for the test masses, and input optics to stabilize the frequency and amplitude of the laser beam. Even with these additions, because of the extremely small size of gravitational wave amplitudes, gravitational waveforms continue to be buried within the noise of the detector. To extract the signal, a technique called, matched filtering is employed. Matched filtering is where a template of a signal is integrated in time over the data set. If the signal is in the data set, though it is buried in noise, integrating a template of the signal over it extracts the signal from the noise. The generation of signal templates to extract gravitational waveforms is the arena of numerical relativity.

4 Numerical Relativity

In General Relativity, space and time are not independent but are sewn together into a four dimensional manifold called spacetime. There is no distinction between time and space when the Einstein field equations are in their fully covariant form. While this is a powerful concept, it is not very useful when one wishes to simulate these equations on a computer. In computer simulations, time is a special direction. To simulate a system, it is prescribed appropriate initial data and then evolved "forward" in time. Therefore, to transform the Einstein field equations into something that can be simulated, spacetime must be split into time and space parts [15, pgs. 505-516]. This can be accomplished by slicing spacetime using a foliation scheme. Foliation schemes that can be used slice spacetime into space-like hypersurfaces [22], null (light-like) hypersurfaces [23, 24], or some hybrid of the two. For example, the method of *Cauchy characteristic matching* [24, 25, 26], or *asymptotically null hypersurfaces* [27]. The most common choice of spacetime foliation is to use space-like hypersurfaces. This approach is generally know as 3 + 1 relativity. In this work however, we will foliate spacetime using null hypersurfaces.

4.1 3 + 1 Relativity

The Einstein field equations in their fully covariant form do not form an evolution system. This is because the equations do not contain second derivatives with respect to the evolution variable t of the spacetime metric $g_{\alpha\beta}$. Using the Bianchi identity, we can show this explicitly [15, pgs. 505-516]. Since the Einstein tensor $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$ obeys the Bianchi identity

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{4.1}$$

we can obtain the identity

$$G^{\alpha t}{}_{;t} = -(G^{\alpha x^1}{}_{;x^1} + G^{\alpha x^2}{}_{;x^2} + G^{\alpha x^3}{}_{;x^3}). \tag{4.2}$$

Now, $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$ and $R_{\alpha\beta} = R^{\gamma}{}_{\alpha\gamma\beta}$, where $R^{\delta}{}_{\alpha\gamma\beta}$ has the form $\partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$ and the connection coefficients Γ have the form $g^{-1}\partial g$. We see that $G^{\alpha\beta}$ contains up to second derivatives in time and space. The right side of equation (4.2) contains third derivatives, but these are not with respect to t. Since equation (4.2) is an identity, this implies that $G^{\alpha t}$ can only depend on $g_{\alpha\beta}$, $g_{\alpha\beta,p}$, $g_{\alpha\beta,t}$ $g_{\alpha\beta,pq}$, $g_{\alpha\beta,pt}$, $g_{\alpha\beta,p}$. Therefore, the equations $G^{\alpha t} = 8\pi T^{\alpha t}$ contain no second derivatives with respect to t of $g_{\alpha\beta}$. They only contain up to first derivatives with respect to t of $g_{\alpha\beta}$, meaning that the equations $G^{\alpha t} = 8\pi T^{\alpha t}$ constrain the initial data. The program of 3 + 1 relativity is to transform the Einstein field equations into an evolution system by splitting spacetime into three space dimensions and one time dimension. The splitting transforms the smooth four-dimensional manifold of spacetime into an infinite decker sandwich of infinitesimally thin slices. Each slice represents an instant in time. They are the level surfaces of t.The Einstein field equations are evolved "forward" in time by projecting the equations onto the level surface $t = 0$ and then evolving this surface to the other successive surfaces $t = \delta t_1$, $t = \delta t_2$ $t = \delta t_3, ...,$ $t = \delta t_n$. Projecting the Einstein field equations onto the level surface $t = 0$ is not a trivial task. In section 1.1.1, the basic tools to project the field equations are introduced and in section 1.1.2, the same tools are used to project the equations. In section 1.1.3, we discuss a limitation of 3 + 1 relativity.

4.1.1 Space-like Hypersurface Foliation

The level surfaces of t are three dimensional space-like hypersurfaces. For convenience, these hypersurfaces are denoted by $\sum_t$. Points on $\sum_t$ are designated by the space coordinates x^1, x^2, and x^3. Thus, the coordinates of spacetime are (t, x^1, x^2, x^3). The general form of the spacetime metric in 3 + 1 relativity is [28, pg. 30]

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \tag{4.3}$$

where N is the lapse function and measures the proper time of an observer traveling normal to the hypersurfaces $\sum_t$; β^i is the shift vector and accounts for the shifting of the space coordinates (x^1, x^2, x^3); and γ_{ij} is the metric that is intrinsic to each space-like hypersurface. There are three important quantities that will enable us to write the Einstein field equations as an evolution system. These are: the vector normal to $\sum_t$, a tensor that projects component objects onto $\sum_t$, and the extrinsic curvature tensor $K_{\alpha\beta}$ of $\sum_t$. Since $\sum_t$ are level surfaces of t, the gradient of t will be normal to the surface. Choosing n^μ to be a future pointing time-like unit normal of the normal traveling observer, we have [28, pg. 30]

$$n^\mu = -N\nabla^\mu t. \tag{4.4}$$

To see that equation (4.4) is a future pointing, time-like, unit normal vector, contract n^μ with itself:

$$\begin{aligned} n^\mu n_\mu &= N^2(\nabla^\mu t \nabla_\mu t) = N^2(g^{\mu\beta}\nabla_\beta t \nabla_\mu t) = N^2 g^{\mu\beta}(dt)_\beta (dt)_\mu \\ &= N^2 g^{00}(dt)_0(dt)_0 = N^2(-N^{-2}) = -1. \end{aligned} \tag{4.5}$$

Now, choosing γ^λ_ν to be the projection tensor we have [14, pg. 29]

$$\gamma^\lambda_\nu = \delta^\lambda_v + n^\lambda n_\nu. \tag{4.6}$$

It is seen that equation (4.6) has the desired projection properties by operating γ^λ_ν on n^μ and t^μ :

$$\gamma^\lambda_\nu n^\nu = \delta^\lambda_v n^\nu + n^\lambda n_\nu n^\nu = n^\lambda - n^\lambda = 0 \tag{4.7(a)}$$

$$\gamma^\lambda_\nu t^\nu = \delta^\lambda_v t^\nu + n^\lambda n_\nu t^\nu = t^\lambda. \tag{4.7(b)}$$

Now, the *extrinsic curvature tensor* $K_{\alpha\beta}$, an object that qualifies how the hypersurfaces $\sum_t$ are embedded in spacetime, can be written as [14, pg. 24]

$$K(u,v) = -u \cdot \nabla_v n, \tag{4.8}$$

where the vectors u and v are tangent to $\sum_t$ and n is the normal vector given in equation (4.4). The components $K_{\alpha\beta}$ of the extrinsic curvature tensor are

$$\begin{aligned} K_{\alpha\beta} &= g(e_\alpha, \nabla_{e_\beta} n^\gamma e_\gamma) = g(e_\alpha, (\nabla_{e_\beta}(n^\gamma)e_\gamma + n^\gamma \nabla_{e_\beta} e_\gamma) \\ &= g(e_\alpha, (e_\beta(n^\gamma)e_\gamma + n^\gamma e_\delta \Gamma^\delta{}_{\gamma\beta})) = g(e_\alpha, (e_\beta(n^\gamma)e_\gamma + n^\delta e_\gamma \Gamma^\gamma{}_{\delta\beta})) \\ &= (e_\beta(n^\gamma) + n^\delta \Gamma^\gamma{}_{\delta\beta}) g(e_\alpha, e_\gamma) v = (e_\beta(n^\gamma) + n^\delta \Gamma^\gamma{}_{\delta\beta}) g_{\alpha\gamma.} \end{aligned} \tag{4.9}$$

In terms of the intrinsic three dimensional metric γ_{ij} on $\sum_t$, the extrinsic curvature tensor is [28, pg. 30]

$$K_{ij} = -\frac{1}{2} \pounds_n \gamma_{ij}, \tag{4.10}$$

where $\pounds_n$ is the Lie derivative with respect to the normal vector.

Constraint and Evolution Equations In order to write the Einstein field equations as an evolution system, the field equations must be projected onto the hypersurfaces $\sum_t$. This is accomplished via the Gauss, Codazzi, and Ricci equations. These equations decompose the spacetime Riemann tensor into objects on the hypersurfaces $\sum_t$, namely the connection D associated with γ_{ij}, the extrinsic curvature tensor $K_{\alpha\beta}$, and the Riemann tensor on the hypersurfaces $\sum_t$. Let

$(R^3)^{\alpha}{}_{\beta\gamma\delta}$ denote the Riemann tensor on the hypersurfaces $\sum_t$. The Gauss, Codazzi, and Ricci equations are obtained by contracting the projection tensor γ^{λ}_{ν}, the normal vector n^{μ}, and the metric $\gamma_{\alpha\beta}$ a sufficient number of times with the spacetime Riemann tensor. To obtain the Gauss equation, the projection tensor must be contracted with all four of the indices of $R^{\alpha}{}_{\beta\gamma\delta}$ [14, pg. 35]:

$$\gamma^{\lambda}_{\alpha}\gamma^{\beta}_{\upsilon}\gamma^{\gamma}_{\epsilon}\gamma^{\delta}_{\eta}R^{\alpha}{}_{\beta\gamma\delta} = (R^3)^{\lambda}{}_{\upsilon\epsilon\eta} + K^{\lambda}{}_{\epsilon}K_{\upsilon\eta} - K^{\lambda}{}_{\eta}K_{\epsilon\upsilon}. \tag{4.11}$$

To obtain the Codazzi equation, the projection tensor must be contracted on the first, third, and fourth indices of $R^{\alpha}{}_{\beta\gamma\delta}$, while the normal vector is contracted on the second indice [14, pg. 36]:

$$\gamma^{\lambda}_{\alpha}n^{\beta}\gamma^{\gamma}_{\epsilon}\gamma^{\delta}_{\eta}R^{\alpha}{}_{\beta\gamma\delta} = D_{\eta}K^{\lambda}{}_{\epsilon} - D_{\epsilon}K^{\lambda}{}_{\eta}. \tag{4.12}$$

The Ricci equation is obtained by lowering the first index of $R^{\alpha}{}_{\beta\gamma\delta}$ using $\gamma_{\lambda\alpha}$, contracting the projection tensor with the third indice, and then contracting the normal vector with the second and fourth indices [14, pg. 48]:

$$\gamma_{\lambda\alpha}n^{\beta}\gamma^{\gamma}_{\epsilon}n^{\delta}R^{\alpha}{}_{\beta\gamma\delta} = \pounds_{n}K_{\lambda\epsilon} + \frac{1}{N}D_{\lambda}D_{\epsilon}N + K_{\lambda\alpha}K^{\alpha}{}_{\epsilon}. \tag{4.13}$$

Using equations (4.11), (4.12), and (4.13) the Einstein field equations (for the vacuum case) can be decomposed into the system of equations [14, pgs. 53 - 54]

$$\pounds_{n}K_{ij} = ((R^3)_{ij} + KK_{ij} - 2K_{ik}K^{k}{}_{j}) - \frac{1}{N}D_{i}D_{j}N \tag{4.14}$$

$$(R^3) + K^2 - K_{ij}K^{ij} = 0 \tag{4.15}$$

$$D_j(K^{ij} - \gamma^{ij}K) = 0. \tag{4.16}$$

Now, equations (4.14-4.16) and (4.9) are the Einstein field equations in the form of an evolution system. The initial data of this system are the extrinsic curvature tensor K_{ij} and the metric γ_{ij} intrinsic to the hypersurfaces $\sum_t$.

4.1.2 Artificial Outer Boundary

Since a computer is a finite machine, the computational grid contains a finite number of points. It is impossible to follow a wave out to infinity because it would require an infinite number of spatial grid points. The utilization of space-like hypersurfaces to study gravitational waves imposes an artificial outer boundary, since the gravitational waves can only be evolved to some finite distance r from the system. Key problems of having this artificial boundary are the reflection of waves at the boundary and the extraction of gravitational wave forms at a finite distance from the system.

4.2 Null Hypersurfaces

In $3 + 1$ relativity, each hypersurface representing time t is accomplished by projecting the Einstein equations onto the space-like hypersurface $t = 0$ and then evolving this hypersurface to other space-like hypersurfaces $t = \delta t_1$, $t = \delta t_2$ $t = \delta t_3, ..., t = \delta t_n$. In contrast, the foliation of spacetime by null hypersurfaces is accomplished by transforming the spacetime metric into null coordinates. A potential problem of using null hypersurfaces is the formation of caustics. One strategy that avoids the formation of caustics is confining the system to a central world tube, while the surrounding spacetime becomes asymptotically flat as infinity (I "scri") is approached. In the world tube, spacetime is foliated by space-like hypersurfaces. Spacetime external to the world tube is foliated by null hypersurfaces. The foliations of the world tube and spacetime external to the world tube are matched at the world tube boundary. This foliation method is know as *Cauchy characteristic matching*. When studying gravitational radiation, foliating spacetime using null hypersurfaces offers a distinct advantage over foliating spacetime via space-like hypersurfaces. Null hypersurface foliation introduces no artificial outer boundary, since each null hypersurface extends to future null infinity (I^+ "scri +") and gravitational radiation emitted at time t travels along the corresponding null hypersurface to future null infinity. In the computer, it seems that an artificial cut off still has to be introduced, since marching to I^+ along a null hypersurface requires an infinite number of spatial grid points. However, this is not necessary, since I^+ can be brought to a finite distance away from the system emitting gravitational waves using a technique called *compactification* [29]. If the given spacetime is asymptotically flat, interesting quantities pertaining to the mass of the system and how much energy is carried away by gravitational radiation can be formed. These quantities are know as the *Bondi mass* and *Bondi mass loss* (news function) [23, 30, 31].

4.2.1 Null Coordinates

Null coordinates are used to label light rays. A *physical picture* is that of a candle at the origin of a spherical coordinate system. Let u represent all light rays emanating from the candle at time t; let r point along each light ray radiated at time t; and the polar angle θ and the the azimuthal angle ϕ label positions for each light ray. A metric in null coordinates in this spherical spacetime could be

$$ds^2 = -Adu^2 + Bdudr + Cdud\theta + Dd\theta^2 + Ed\phi^2, \tag{4.17}$$

where the cross terms ensure that r and θ point along light rays. An example of null coordinates, only involving u, is Minkowski space with one spatial dimension. Letting t label the time axis and x the space axis, the spacetime metric is

$$ds^2 = -dt^2 + dx^2. \tag{4.18}$$

Since light rays traveling to the right move along the lines $t - x = const.$, then $u = t - x$ labels all light rays traveling to the right radiated at time t. Transforming the above metric into this null coordinate yields

$$ds^2 = -(du^2 + 2dudx). \tag{4.19}$$

Introducing a second null coordinate, $v = t + x$, for light rays moving to the left, reduces the metric in equation (17) to the simple form

$$ds^2 = -dvdu. \tag{4.20}$$

A more complicated example of a spacetime in null coordinates is the *Bondi-Sachs metric* [23]

$$\begin{aligned} ds^2 &= (\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma})du^2 + 2e^{2\beta}dudr + 2Ur^2e^{2\gamma}dud\theta \\ &\quad -r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2). \end{aligned} \tag{4.21}$$

Motivation for using null coordinates is that problems involving radiation (electromagnetic or gravitational) are simplified. Consider the one dimensional vibrating string in flat space. In typical $t - x$ coordinates, the equation for the vibrating string is

$$-\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial^2\Phi}{\partial x^2} = g^{\alpha\beta}(\Phi_{,\alpha})_{;\beta} = 0, \tag{4.22}$$

where Φ has the form $\Phi = f(t-x) + g(t+x)$. In terms of the null coordinate $u = t - x$, the wave equation for Φ becomes

$$\frac{\partial^2\Phi}{\partial u\partial x} - \frac{1}{2}\frac{\partial^2\Phi}{\partial x^2} = 0. \tag{4.23}$$

The general solution to this equation is $\Phi = f(u) + g(u + 2x)$. The solution $f(u)$ represents an outward traveling wave, while $g(u + 2x)$ represents an inward traveling wave. If we only consider outgoing waves, which is what we would think of, if we were dealing with light rays or gravitational waves emanating from a source, the coordinate u labels each segment of an outgoing wave. A null coordinate for incoming waves can also be introduced. If $v = t + x$, then $g(u + 2x) = g(t + x) = g(v)$. The null coordinate v labels each segment of an incoming wave. Using null coordinates u and v, the one-dimensional wave equation takes the simple form

$$\frac{\partial^2 \Phi}{\partial v \partial u} = 0. \tag{4.24}$$

The general solution is $\Phi = f(u) + g(v)$.Thus, in the right coordinates, the one dimensional wave equation reduces to a partial differential equation with one term. Wave equations are not the only thing simplified when null coordinates are used. The Bondi-Sachs metric represents a vacuum spacetime with axisymmetry. The metric is in terms of outgoing null coordinates. When the Einstein tensor is calculated, evolution and hypersurface equations, in terms of the functions $U(u, r, \theta)$, $V(u, r, \theta)$, $\beta(u, r, \theta)$ and $\gamma(u, r, \theta)$, are formed. From [23] these equations, in terms of Ricci tensor components, are

$$0 = R_{11} = -4[\beta_{,r} - \frac{1}{2}r(\gamma_{,r})^2]r^{-1} \tag{4.25(a)}$$

$$\begin{aligned} 0 &= -2r^2 R_{12} = [r^4 e^{2(\gamma-\beta)} U_{,r}]_r - 2r^2[\beta_{,r\theta} - \gamma_{,r\theta} + 2\gamma_{,r}\gamma_{,\theta} \\ &\quad -2\beta_{,\theta} r^{-1} - 2\gamma_{,r} \cot\theta] \end{aligned} \tag{4.25(b)}$$

$$\begin{aligned} 0 &= R_{22} e^{2(\beta-\gamma)} - r^2 R^3_3 e^{2\beta} = 2V_{,r} + \frac{1}{2} r^4 e^{2(\beta-\gamma)} (U_{,r})^2 \\ &\quad -r^2 U_{,r\theta} - 4rU_{,\theta} - r^2 U_{,r}\cot\theta - 4rU\cot\theta + 2e^{2(\beta-\gamma)}[-1 \\ &\quad -(3\gamma_{,\theta} - \beta_{,\theta})\cot\theta - \gamma_{,\theta\theta} + \beta_{,\theta\theta} + (\beta_{,\theta})^2 + 2\gamma_{,\theta}(\gamma_{,\theta} - \beta_{,\theta})] \end{aligned} \tag{4.25(c)}$$

$$
\begin{aligned}
0 \;=\; & -R_3^3 e^{2\beta} r^2 = 2r(r\gamma)_{,ur} + (1 - r\gamma_{,r})V_{,r} - (r\gamma_{,rr} + \gamma_{,r})V \quad (4.25(\mathrm{d}))\\
& -r(1 - r\gamma_{,r})V_{,\theta} - r^2(\cot\theta - \gamma_{,\theta})U_{,r}\\
& r(2r\gamma_{,r\theta} + 2\gamma_{,\theta} + r\gamma_{,r}\cot\theta - 3\cot\theta)U\\
& e^{2(\beta-\gamma)}[-1 - (3\gamma_{,\theta} - 2\beta_{,\theta})\cot\theta - \gamma_{,\theta\theta} + 2\gamma_{,\theta}(\gamma_{,\theta} - \beta_{,\theta})]
\end{aligned}
$$

Upon inspection, it is seen that this system of equations forms a hiearchy. Prescribing initial data β, equation R_{11} can be solved for γ. Inserting β and γ into $-2r^2R_{12}$ allows for the solution of U. Finally, the insertion of β, γ, and U into $R_{22}e^{2(\beta-\gamma)} - r^2R_3^3e^{2\beta}$ allows one to solve for V. The equation $-R_3^3e^{2\beta}r^2$ can be solved for $\gamma_{,ur}$ allowing one to find β, U, and V on a new hypersurface. Unlike the evolution system formed using space-like hypersurfaces, where the initial data must satisfy elliptic constraint equations, the initial data of this system is free and the evolution from one hypersurface to a new hypersurface is performed by solving a hiearchal sequence of equations. Another example of a spacetime in null coordinates is a spherically symmetric spacetime with stress-energy tensor

$$T_{\mu\nu} = \nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\Phi\nabla^\alpha\Phi \tag{4.26}$$

for a zero rest mass scalar field Φ and metric [30]

$$ds^2 = e^{2\beta}(\frac{V}{r}du^2 + 2dudr) - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.27}$$

where β and V are functions of u and r. Calculating the Einstein tensor

$$G_{\mu\nu} = 8\pi T_{\mu\nu}[\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}\nabla_\alpha\Phi\nabla^\alpha\Phi] \tag{4.28}$$

yields equations in terms β and V. After several manipulations, the field equations are found to be equivalent to

$$\beta_{,r} = 2\pi r(\Phi_{,r})^2 \tag{4.29}$$

$$V_{,r} = e^{2\beta}. \tag{4.30}$$

Again, a hiearchy of equations is formed; however, in this case the initial data Φ is not free, it must satisfy the scalar wave equation

$$\Box\Phi = 2r\Phi_{,ur} - \frac{(rV\Phi_{,r})_{,r}}{r} = 0. \tag{4.31}$$

An example of a non-spherically symmetric spacetime in null coordinates is a spacetime with a two-parameter space-like isometry group [32]:

$$ds^2 = -e^{2a}(du^2 + 2dudr) + R(e^{-2\psi}d\varphi^2 + e^{2\psi}dz^2). \tag{4.32}$$

This metric is in null form because the two-parameter space-like isometry group is exploited by causing an observer to travel in a fixed φ and z direction. Calculating the Einstein tensor yields equations in terms of R, ψ, and a. Several manipulations later, two evolution equations and one hypersurface equation yield:

$$0 = 2e^{2(a+\psi)}\left(R_{zz} + R_{\phi\phi}\right) = R_{,ur} - \frac{1}{2}R_{,rr} \tag{4.33}$$

$$0 = 2e^{-2(a+\psi)}\left(R_{zz} - R_{\phi\phi}\right) = \psi_{,ur} - \frac{1}{2}\psi_{,rr} \tag{4.34}$$

$$-\psi_{,r}\frac{(R_{,r} - R_{,u})}{R} + \psi_{,u}\frac{R_{,r}}{R} \tag{4.35}$$

$$0 = 2R^2R_{rr} = R_{,r}^2 - 4R^2\psi_{,r}^2 + R(4a_{,r}R_{,r} - 2R_{,rr})$$

A general inference from these examples is that spacetimes in null coordinates form hiearchies of equations for the unknown functions in the metric. In the axisymmetric case, the initial data is free. In the spherically symmetric case with stress-energy tensor for a massless scalar field and for the cylindrically symmetric case, the initial data must satisfy a wave equation.

4.2.2 Compactification

The ideal behind compactification is that there exists a smooth scalar function Ω and the physical metric $g_{\alpha\beta}$ is mapped to an unphysical metric $g'_{\alpha\beta}$, where in the unphysical spacetime I is at finite points. The mapping $\Omega : g_{\alpha\beta} \rightarrow g'_{\alpha\beta}$ is given by

$$\Omega^2 g_{\alpha\beta} = g'_{\alpha\beta}. \tag{4.36}$$

An example of spacetime compactification is best given by considering Minkowski spacetime. The metric for Minkowski spacetime is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{4.37}$$

The goal is to transform the Minkowski metric into a coordinate frame. Therefore, we have the mapping

$$(-\infty, \infty) \rightarrow [a, b] \tag{4.38}$$

between the coordinates of the frame and new coordinates. A smooth function Ω^2 can be factored out of the metric. The first step is to transform the Minkowski metric into spherical coordinates

$$\begin{aligned} x &= r\sin(\theta)\cos(\phi) \\ y &= r\sin(\theta)\sin(\phi) \\ z &= r\cos(\theta). \end{aligned} \tag{4.39}$$

This transformation yields

$$ds^2 = -dt^2 + dr^2 + r^2 d\sigma^2, \tag{4.40}$$

where $d\sigma^2$ is the metric of the unit sphere. The next step is to transform the above metric into the advanced and retarded null coordinates $v = t + r$ and $u = t - r$. This yields

$$ds^2 = -\frac{1}{2}dvdu + \frac{1}{4}(v-u)^2 d\sigma^2. \tag{4.41}$$

The ranges of u and v are $-\infty \leq u \leq \infty$ and $-\infty \leq v \leq \infty$, meaning that u and v can be represented as

$$u = \tan(U) \tag{4.42(a)}$$

$$v = \tan(V), \tag{4.42(b)}$$

where U and V have the ranges $-\frac{\pi}{2} \leq U \leq \frac{\pi}{2}$ and $-\frac{\pi}{2} \leq V \leq \frac{\pi}{2}$. Transforming the metric in equation (4.41) into these new coordinates yields

$$\begin{aligned} ds^2 &= \frac{1}{4\cos^2(V)\cos^2(U)}(-2dV\,dU + (\cos^2(U) + \cos^2(V) \\ &\quad -2\cos(V)\cos(U)\cos(V-U)d\sigma^2). \end{aligned} \tag{4.43}$$

We can see that the metric in these new coordinates is not defined at $U = \pm\frac{\pi}{2}$ or $V = \pm\frac{\pi}{2}$. However, if a different metric g' is defined to be conformally related to the Minkowski metric g by the relation $g' = \Omega^2 g$ where $\Omega = 2\cos(V)\cos(U)$, we see that

$$g' = -2dVdU + (\cos^2(U) + \cos^2(V) - 2\cos(V)\cos(U)\cos(V-U)d\sigma^2. \quad (4.44)$$

The conformal metric g' is finite at $U = \pm\frac{\pi}{2}$ and $V = \pm\frac{\pi}{2}$.Since I corresponds to $U = \pm\frac{\pi}{2}$ and $V = \pm\frac{\pi}{2}$, we have effectively mapped spacetime onto a finite grid. Any simulation in the conformal spacetime using finite differencing can be carried all the way out to I, since I can be reached by a finite number of grid points.

4.2.3 Constraint and Evolution equations

In $3+1$ relativity, a normal vector is constructed and the Einstein equations are then projected onto the space-like hypersurfaces. As a result, the initial data has to satisfy elliptic constraint equations. If spacetime is foliated using null hypersurfaces and a normal vector is constructed to project the Einstein equations onto the null hypersurfaces, we are immediately confronted with a peculiarity. Vectors normal to a null hypersurface are also tangent to the hypersurface. Hence, the Einstein equations and normal vector lie in the same hypersurface. A consequence of this situation is that the Einstein equations now do not impose elliptic constraints on the initial data.

5 Massless Scalar Field Evolution

The particular spacetime, which we will be working in, has a metric of the form

$$ds^2 = e^{2a}(-dt^2 + dr^2) + R\left(e^{-2\psi}d\phi^2 + e^{2\psi}dz^2\right). \tag{5.1}$$

This spacetime has a two-parameter space-like isometry group with ϕ and z being the group coordinates. The functions within the metric a, R, and ψ only depend on t and r. The area traced out by each group orbit is proportional to R. The function a is a dilation factor and ψ represents plus-polarized gravitational waves. The equation that governs the evolution of our massless scalar field is the *scalar wave equation*. For a specific choice of parameters, the scalar field can imitate the amplitude behavior of a light ray, or a gravitational wave traveling along a null hypersurface. However, we will first perform a null hypersurface foliation from the view point of an observer traveling in a fixed ϕ and z direction. Then, we will compactify the radial coordinate r and construct equations for R, ψ, and a.

5.1 Null Hypersurface Foliation

If an observer travels in a fixed ϕ and z direction, his world line is given by

$$ds_2^2 = e^{2a}(-dt^2 + dr^2). \tag{5.2}$$

The $-dt^2 + dr^2$ portion of the metric is effectively one-dimensional Minkowski space (r is a radial coordinate). This means $ds_2^2 = e^{2a}(-dt^2 + dr^2)$ is conformally flat, since

$$\widetilde{ds}_2^2 = \Omega^{-2} ds_2^2, \tag{5.3}$$

where $\widetilde{ds}_2^2 = -dt^2 + dr^2$ and the conformal factor is $\Omega = e^a$. Thus, we restrict ourselves to the class of observers that travel in fixed ϕ and z directions, because foliating ds^2 into null hypersurfaces amounts to performing a null cone foliation of one-dimensional Minkowski space. To foliate this space into null hypersurfaces, transform $-dt^2 + dr^2$ into the advanced and retarded time coordinates $v = t + r$ and $u = t - r$. The transformation yields

$$\widetilde{ds}_2^2 = -dt^2 + dr^2 = -dvdu. \tag{5.4}$$

Alternatively, we can choose to foliate one-dimensional Minkowski space into either ingoing or outgoing null hypersurfaces by transforming $\widetilde{ds}_2^2$ into either the advanced time coordinate $v = t + r$, or the retarded time coordinate $u = t - r$. In terms of ingoing null hypersurfaces, $\widetilde{ds}_2^2$ is

$$\widetilde{ds}_2^2 = -dv^2 + 2dvdr. \tag{5.5}$$

In terms of outgoing null hypersurfaces, $\widetilde{ds}^2$ is

$$\widetilde{ds}_2^2 = -\left(du^2 + 2dudr\right). \tag{5.6}$$

For the evolution of the massless scalar field, we will use the outgoing null hypersurface foliation. Hence, for our restricted class of observers, the outgoing null hypersurface foliation of our spacetime yields

$$ds^2 = -e^{2a}\left(du^2 + 2dudr\right) + R\left(e^{-2\psi}d\phi^2 + e^{2\psi}dz^2\right). \tag{5.7}$$

5.2 Radial Coordinate Compactification

Anything that travels at the speed of light (massless scalar field, gravitational waves, light, etc.,...) is completely confined within a null hypersurface. In our spacetime, the retarded time coordinate u labels a progression of outgoing null hypersurfaces. Each hypersurface u contains all gravitational waves ψ and the massless scalar field generated at time u at $r = 0$. The gravitational waves and the massless scalar filed travel on each outgoing hypersurface u from $r = 0$ to infinity. It is not possible to evolve on a computer the massless scalar field or the gravitational waves ψ contained within a outgoing null hypersurface u from $r = 0$ to infinity using the radial coordinate r. This would require an infinite amount of memory. However, we can circumnavigate this shortcoming by compactifying the radial coordinate. To compactify the radial coordinate r, let L be a radial distance related to r by the transformation

$$L = \arctan(r). \tag{5.8}$$

Since r has the range $0 \leq r < \infty$, this transformation maps the interval $[0, \infty)$ into $[0, \frac{\pi}{2}]$. Transforming the metric in equation (5.7) into the compactified radial coordinate L yields

$$ds^2 = -e^{2a}\left(du^2 + \frac{2}{\cos^2(L)}dudL\right) + R\left(e^{-2\psi}d\phi^2 + e^{2\psi}dz^2\right). \tag{5.9}$$

The coordinates of each constant ϕ and constant z slice have the range $-\infty < u < \infty$ and $0 \leq L \leq \frac{\pi}{2}$. The metric in equation (5.9) is the background in which we will evolve the massless scalar field.

5.3 Equations for a, R, and ψ

Equations for a, R, and ψ are found by calculating the Einstein tensor components and then forming suitable linear combinations. Since we are working with a vacuum spacetime, only the Ricci tensor components need to be calculated. In vacuum, they are equivalent to the Einstein tensor components. The Ricci tensor components, calculated using Mathematica 7.0 [33], are

$$\begin{aligned} R_{uu} &= \frac{1}{2R^2}(R_{,u}^2 + 2R^2(\cos^4(L)(a_{,LL} - 2\tan(L)a_{,L}) \\ &\quad -2(\psi_{,u}^2 + \cos^2(L)a_{,uL})) + 2R(\cos^2(L)a_{,L}(\cos^2(L)R_{,L} - R_{,u}) \\ &\quad -\cos^2(L)a_{,u}R_{,L} + 2a_uR_u - R_{,uu})) = 0, \end{aligned} \tag{5.10(a)}$$

$$\begin{aligned} R_{Lu} &= \cos^4(L)(a_{,LL} - 2\tan(L)a_{,L}) + \frac{\cos^2(L)R_{,L}R_{,u}}{2R^2} \\ &\quad -2\cos^2(L)(\psi_{,L}\psi_{,u} + a_{,uL}) + \frac{\cos^2(L)}{R}(\cos^2(L)a_{,L}R_{,L} - R_{,uL}) = 0, \end{aligned} \tag{5.10(b)}$$

$$
\begin{aligned}
R_{LL} &= \frac{\cos^4(L)}{2R^2}(R_{,L}^2 - 4R^2\psi_{,L}^2 + R(4a_{,L}R_{,L} \\
&\quad -2\left(R_{,LL} - 2\tan(L)\,R_{,L}\right))) = 0,
\end{aligned} \tag{5.10(c)}
$$

$$
\begin{aligned}
R_{\phi\phi} &= \frac{1}{2}e^{-2(a+\psi)}\cos^2(L)\,(2R_{,L}(\cos^2(L)\,\psi_{,L} - \psi_{,u}) \\
&\quad -\cos^2(L)\,(R_{,LL} - 2\tan(L))R_{,L} + 2(R_{,uL} - \psi_{,L}R_{,u} \\
&\quad +R(\cos^2(L)\,(\psi_{,LL} - 2\tan(L)\,\psi_{,L}) - 2\psi_{,uL}))) = 0,
\end{aligned} \tag{5.10(d)}
$$

$$
\begin{aligned}
R_{zz} &= \frac{1}{2}e^{-2(a+\psi)}\cos^2(L)\,(2(R_{,uL} + \psi_{,L}R_{,u} \\
&\quad -R(\cos^2(L)\left(\psi_{,LL} - 2\tan(L)\,\psi_{,L}\right) - 2\psi_{,uL})) \\
&\quad -2R_{,L}(\cos^2(L)\,\psi_{,L} - \psi_{,u}) - \cos^2(L)\,(R_{,LL} - 2\tan(L)\,R_{,L})) = 0.
\end{aligned} \tag{5.10(e)}
$$

Since the area function R, and the gravitational wave factor ψ modify the ϕ and z directions, equations for R and ψ are found through the linear combination

$$
aR_{\phi\phi} + bR_{zz} = 0. \tag{5.11}
$$

Choosing $a = 2$, and $b = 2$ gives an equation for $R:$

$$
\begin{aligned}
&2R_{zz} + 2R_{\phi\phi} = e^{-2(a+\psi)}\cos^2(L)\,(R_{,uL} \\
&-\frac{1}{2}\cos^2(L)\,(R_{,LL} - 2\tan(L)\,R_{,L})) = 0.
\end{aligned} \tag{5.12}
$$

To ensure $2R_{zz} + 2R_{\phi\phi}$ always equals zero it must be that

$$\begin{aligned} 2e^{2(a+\psi)}\cos^{-2}(L)\left(R_{zz}+R_{\phi\phi}\right) &= \\ R_{,uL}-\frac{1}{2}\cos^{2}(L)\left(R_{,LL}-2\tan(L)\,R_{,L}\right)) &= 0. \end{aligned} \tag{5.13}$$

Choosing $a = 2$, and $b = -2$ gives an equation for ψ :

$$\begin{gathered} 2R_{zz}-2R_{\phi\phi}=e^{-2(a+\psi)}\cos^{2}(L)\,(\psi_{,uL} \\ -\frac{1}{2}\cos^{2}(L)\left(\psi_{,LL}-2\tan(L)\,\psi_{,L}\right) \\ -\frac{\left(\cos^{2}(L)\,R_{,L}-R_{,u}\right)}{2R}\psi_{,L}+\frac{R_{,L}}{2R}\psi_{,u})=0. \end{gathered} \tag{5.14}$$

Likewise, for the R equation to ensure $2R_{zz} - 2R_{\phi\phi}$ always equals zero it must be that

$$\begin{gathered} 2e^{2(a+\psi)}\cos^{-2}(L)\left(R_{zz}-R_{\phi\phi}\right)= \\ \psi_{,uL}-\frac{1}{2}\cos^{2}(L)\left(\psi_{,LL}-2\tan(L)\,\psi_{,L}\right) \\ -\frac{\left(\cos^{2}(L)\,R_{,L}-R_{,u}\right)}{2R}\psi_{,L}+\frac{R_{,L}}{2R}\psi_{,u})=0. \end{gathered} \tag{5.15}$$

The dilation factor a modifies the time direction and the radial distance along each outgoing null hypersurface. This implies that an equation for a is found through the linear combination

$$cR_{uu} + fR_{Lu} = 0. \tag{5.16}$$

Choosing $c = 0$ and $f = 1$ yields equation (5.10(b))

$$\begin{aligned} R_{Lu} &= \cos^4(L)\left(a_{,LL} - 2\tan(L)\, a_{,L}\right) + \frac{\cos^2(L)\, R_{,L} R_{,u}}{2R^2} \\ &\quad -2\cos^2(L)\left(\psi_{,L}\psi_{,u} + a_{,uL}\right) + \frac{\cos^2(L)}{R}\left(\cos^2(L)\, a_{,L} R_{,L} - R_{,uL}\right). \end{aligned}$$

Dividing by $-2\cos^2(L)$ yields

$$\begin{aligned} a_{,uL} - \frac{1}{2}\cos^2(L)\, a_{,LL} + \cos^2(L)\tan(L)\, a_{,L} - \cos^2(L)\frac{R_{,L}}{2R} a_{,L} \\ + \frac{R_{,uL}}{2R} - \frac{R_{,L}R_{,u}}{4R^2} + \psi_{,L}\psi_{,u} = 0. \end{aligned} \tag{5.17}$$

Hence, the equations that govern R, ψ, and a are given by (5.13), (5.15), (5.17), and the remaining Ricci tensor component R_{LL} (5.10(c)). equations (5.13) and (5.15) are evolution equations, while equation (5.10(c)) is a hypersurface equation [34]. Equation (5.17) is a subsidiary equation [24, pg. 12]. Any choice of a, R, and ψ must be consistent with these equations.

5.4 Scalar Wave equation

Our massless scalar field Φ is governed by the homogenous scalar wave equation

$$\nabla^2 \Phi = 0. \tag{5.18}$$

In Minkowski space and in terms of rectangular coordinates, the explicit form of this equation is

$$\nabla^2 \Phi = -\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \tag{5.19}$$

Since our spacetime is $ds^2 = -e^{2a}\left(du^2 + \frac{2}{\cos^2(L)}dudL\right) + R\left(e^{-2\psi}d\phi^2 + e^{2\psi}dz^2\right),$ the equation for our scalar field will have a substantially different form. To find $\nabla^2\Phi = 0$ in ds^2, write it as

$$\nabla \cdot \nabla\Phi = 0. \tag{5.20}$$

Recalling that the metric of a spacetime gives the scalar product of vectors and forms within that spacetime, the above equation may be written as

$$\begin{aligned} g^{-1}\left(\nabla, \nabla\Phi\right) = g^{\alpha\beta}\nabla_\alpha\left(\nabla\Phi\right)_\beta = g^{\alpha\beta}\left(\Phi_{,\beta}\right)_{;\alpha} \\ = g^{\alpha\beta}\left(\Phi_{,\beta\alpha} - \Phi_{,\lambda}\Gamma^\lambda{}_{\beta\alpha}\right) \\ = g^{\alpha\beta}\left(\Phi_{,\beta\alpha} - \Phi_{,u}\Gamma^u{}_{\beta\alpha} - \Phi_{,L}\Gamma^L{}_{\beta\alpha} - \Phi_{,\phi}\Gamma^\phi{}_{\beta\alpha} - \Phi_{,z}\Gamma^z{}_{\beta\alpha}\right) = 0. \end{aligned} \tag{5.21}$$

The inverse metric is

$$g^{\alpha\beta} = \begin{pmatrix} 0 & -e^{-2a}\cos^2(L) & 0 & 0 \\ -e^{-2a}\cos^2(L) & e^{-2a}\cos^4(L) & 0 & 0 \\ 0 & 0 & \frac{e^{2\psi}}{R} & 0 \\ 0 & 0 & 0 & \frac{e^{-2\psi}}{R} \end{pmatrix}, \tag{5.22}$$

meaning that the required connection coefficients are $\Gamma^u{}_{uL}, \Gamma^u{}_{LL}, \Gamma^u{}_{\phi\phi}, \Gamma^u{}_{zz}, \Gamma^L{}_{uL},$ $\Gamma^L{}_{LL}, \Gamma^L{}_{\phi\phi}, \Gamma^L{}_{zz}, \Gamma^\phi{}_{uL}, \Gamma^\phi{}_{LL}, \Gamma^\phi{}_{\phi\phi}, \Gamma^\phi{}_{zz}, \Gamma^z{}_{uL}, \Gamma^z{}_{LL}, \Gamma^z{}_{\phi\phi},$and $\Gamma^z{}_{zz}$. These connection coefficients are

$$\begin{aligned} \Gamma^u{}_{uL} &= 0 \\ \Gamma^u{}_{LL} &= 0 \\ \Gamma^u{}_{\phi\phi} &= \frac{1}{2}e^{-2(a+\psi)}\cos^2(L)\left(R_{,L} - 2R\psi_{,L}\right) \\ \Gamma^u{}_{zz} &= \frac{1}{2}e^{-2(a-\psi)}\cos^2(L)\left(R_{,L} + 2R\psi_{,L}\right) \end{aligned} \tag{5.23}$$

,

$$
\begin{aligned}
\Gamma^{L}{}_{uL} &= \cos^2(L)\, a_{,L} \\
\Gamma^{L}{}_{LL} &= 2\left(\tan(L) + a_{,L}\right) \\
\Gamma^{L}{}_{\phi\phi} &= \frac{1}{2} e^{-2(a+\psi)} \cos^2(L) \left(R_{,u} - \cos^2(L)\, R_{,L}\right. \\
&\quad \left. + 2R\left(\cos^2(L)\, \psi_{,L} - \psi_{,u}\right)\right) \\
\Gamma^{L}{}_{zz} &= \frac{1}{2} e^{-2(a-\psi)} \cos^2(L) \left(R_{,u} - \cos^2(L)\, R_{,L}\right. \\
&\quad \left. - 2R\left(\cos^2(L)\, \psi_{,L} - \psi_{,u}\right)\right)
\end{aligned} \tag{5.24}
$$

,

$$
\begin{aligned}
\Gamma^{\phi}{}_{uL} &= 0 \\
\Gamma^{\phi}{}_{LL} &= 0 \\
\Gamma^{\phi}{}_{\phi\phi} &= 0 \\
\Gamma^{\phi}{}_{zz} &= 0
\end{aligned} \tag{5.25}
$$

, and

$$
\begin{aligned}
\Gamma^{z}{}_{uL} &= 0 \\
\Gamma^{z}{}_{LL} &= 0 \\
\Gamma^{z}{}_{\phi\phi} &= 0 \\
\Gamma^{z}{}_{zz} &= 0.
\end{aligned} \tag{5.26}
$$

Therefore, the equation for our scalar field within this spacetime is

$$\nabla^2\Phi = -2e^{-2a}\cos^2(L)\left(\Phi_{,uL} - \Phi_{,\lambda}\Gamma^{\lambda}{}_{uL}\right) \tag{5.27}$$

$$+e^{-2a}\cos^4(L)\left(\Phi_{,LL} - \Phi_{,\lambda}\Gamma^{\lambda}{}_{LL}\right)$$

$$+\frac{e^{2\psi}}{R}\left(\Phi_{,\phi\phi} - \Phi_{,\lambda}\Gamma^{\lambda}{}_{\phi\phi}\right) + \frac{e^{-2\psi}}{R}\left(\Phi_{,zz} - \Phi_{,\lambda}\Gamma^{\lambda}{}_{zz}\right)$$

$$= -2e^{-2a}\cos^2(L)\left(\Phi_{,uL} - \cos^2(L)\,a_{,L}\Phi_{,L}\right)$$

$$+e^{-2a}\cos^4(L)\left(\Phi_{,LL} - 2\left(\tan L + a_{,L}\right)\Phi_{,L}\right)$$

$$-\frac{e^{-2a}}{2R}\cos^2(L)\left(R_{,L} - 2R\psi_{,L}\right)\Phi_{,u}$$

$$-\frac{e^{-2a}}{2R}\cos^2(L)\left(R_{,u} - \cos^2(L)\,R_{,L}\right.$$

$$\left.+2R\left(\cos^2(L)\,\psi_{,L} - \psi_{,u}\right)\right)\Phi_{,L}$$

$$-\frac{e^{-2a}}{2R}\cos^2(L)\left(R_{,L} + 2R\psi_{,L}\right)\Phi_{,u}$$

$$-\frac{e^{-2a}}{2R}\cos^2(L)\left(R_{,u} - \cos^2(L)\,R_{,L}\right.$$

$$\left.-2R\left(\cos^2(L)\,\psi_{,L} - \psi_{,u}\right)\right)\Phi_{,L}$$

$$+\frac{e^{2\psi}}{R}\Phi_{,\phi\phi} + \frac{e^{-2\psi}}{R}\Phi_{,zz} = 0$$

Simplifying the above equation yields

$$\nabla^2\Phi = -2e^{-2a}\cos^2(L)\left(\Phi_{,uL} - \cos^2(L)\,a_{,L}\Phi_{,L}\right) \tag{5.28}$$
$$+e^{-2a}\cos^4(L)\left(\Phi_{,LL} - 2\left(\tan(L) + a_{,L}\right)\Phi_{,L}\right)$$
$$-e^{-2a}\frac{R_{,L}}{R}\cos^2(L)\,\Phi_{,u}$$
$$-\frac{e^{-2a}}{R}\cos^2(L)\,(R_{,u} - \cos^2(L)\,R_{,L})\Phi_{,L}$$
$$\frac{e^{2\psi}}{R}\Phi_{,\phi\phi} + \frac{e^{-2\psi}}{R}\Phi_{,zz} = 0.$$

Dividing through by $-2e^{-2a}\cos^2(L)$ and grouping like terms yields

$$\Phi_{,uL} - \frac{1}{2}\cos^2(L)\,\Phi_{,LL} + \frac{R_{,L}}{2R}\Phi_{,u} \tag{5.29}$$
$$+(\cos^2(L)\tan(L) + \frac{(R_{,u} - \cos^2(L)\,R_{,L})}{2R})\Phi_{,L}$$
$$-\frac{e^{2(\psi+a)}}{2R\cos^2(L)}\Phi_{,\phi\phi} - \frac{e^{-2(\psi-a)}}{2R\cos^2(L)}\Phi_{,zz} = 0.$$

Since the spacetime slices of constant ϕ and z have been foliated into null hypersurfaces, we only require Φ to depend on u and L. The scalar wave equation that governs the massless scalar field is then

$$\Phi_{,uL} - \frac{1}{2}\cos^2(L)\,\Phi_{,LL} + \frac{R_{,L}}{2R}\Phi_{,u} \tag{5.30}$$
$$+(\cos^2(L)\tan(L) + \frac{(R_{,u} - \cos^2(L)\,R_{,L})}{2R})\Phi_{,L} = 0.$$

5.5 One Dimensional Waves

In vacuum, when a light ray or a gravitational wave travel outwards from a source along a null hypersurface, they do not diminish in amplitude. If f denotes their amplitude, and u labels outgoing null hypersurfaces, then f will depend on u. If Φ is to imitate this behavior, equation (5.30) must have solutions of the form $\Phi = f(u)$. However, these solutions are possible only if the pure time derivative term of Φ is eliminated. We can eliminate this term by choosing $R =$ constant, or $R = H(u)$.

5.5.1 $R =$ Constant

When we choose $R =$ constant, equation (5.30) becomes

$$\Phi_{,uL} - \frac{1}{2}\cos^2(L)\,\Phi_{,LL} + \cos^2(L)\tan(L)\,\Phi_{,L} = 0. \tag{5.31}$$

By inspection, we see that $\Phi(u, L) = F(u)$ is a solution. Using MAPLE, the general solution is found to be

$$\begin{aligned} \Phi(u, L) &= F(u) \\ &\quad + \int F_2(-(u + 2\tan(L)))e^P dL, \\ P &= \frac{1}{2}\left(\int \sin(2\arctan(-\frac{B}{2} + \frac{u}{2} + \tan(L)))dB\right)]_{B=u}. \end{aligned} \tag{5.32}$$

The additional part to the solution $\Phi(u, L)$ represents incoming waves. Previously, it was stated that a, R, and ψ must be consistent with equations (5.13), (5.15), (5.17), and (5.10(c)). By inspection, we see that $R =$constant satisfies equation

(5.13). For R =constant the equations (5.15), (5.17), and (5.10(c)) become

$$2e^{2(a+\psi)}\cos^{-2}(L)\left(R_{zz}-R_{\phi\phi}\right)= \tag{5.33}$$
$$\psi_{,uL}-\frac{1}{2}\cos^{2}(L)\left(\psi_{,LL}-2\tan(L)\,\psi_{,L}\right)=0,$$

$$a_{,uL}-\frac{1}{2}\cos^{2}(L)\,a_{,LL}+\cos^{2}(L)\tan(L)\,a_{,L}+\psi_{,L}\psi_{,u}=0, \tag{5.34}$$

and

$$R_{LL}=-2\cos^{4}(L)\,\psi_{,L}^{2}=0. \tag{5.35}$$

For $R_{LL}=0$ to always be satisfied, it must be that $\psi=g(u)$, where g is an arbitrary function. It is easy to see that $\psi=g(u)$ satisfies equation (5.33). Now inserting $\psi=g(u)$ into equation (5.34) yields

$$a_{,uL}-\frac{1}{2}\cos^{2}(L)\,a_{,LL}+\cos^{2}(L)\tan(L)\,a_{,L}=0. \tag{5.36}$$

Using MAPLE, the solution to this equation is

$$\begin{aligned} a &= F(u)+\int F_2(-(u+2\tan(L)))e^{N}dL \qquad (5.37)\\ N &= \frac{1}{2}\left(\int -\sin(2\arctan(-\frac{B}{2}+\frac{u}{2}+\tan(L)))dB\right)]_{B=u}. \end{aligned}$$

Hence, when R = constant, equations (5.15), (5.17), and (5.10(c)) are satisfied when $\psi=g(u)$ and $a=F(u)+\int F_2(-(u+2\tan(L)))e^{N}dL$.

5.5.2 $R = H(u)$.

For this particular choice of area function equation (5.30) becomes

$$\Phi_{,uL} - \frac{1}{2}\cos^2(L)\,\Phi_{,LL} + \tag{5.38}$$
$$+(\cos^2(L)\tan(L) + \frac{1}{2H(u)}\frac{dH(u)}{du})\Phi_{,L} = 0.$$

By inspection we see that $\Phi(u, L) = F(u)$ satisfies equation (5.38). To obtain a general solution using MAPLE, we must give an explicit $H(u)$. The general solution has the schematic form $\Phi(u,\, L) = F_1(u)\ +$ *incoming solution,* where the incoming solution part depends on the choice of $H(u)$. However, prescribing an explicit $H(u)$ is not needed since we are not interested in the incoming solution. Now, consider what a and ψ must be so that equations (5.13), (5.15), (5.17), and (5.10(c)) are consistent. By inspection, we see that $R = H(u)$ satisfies equation (5.13). Inserting $R = u$ into equations (5.15), (5.17), and (5.10(c)) yields

$$2e^{2(a+\psi)}\cos^{-2}(L)\,(R_{zz} - R_{\phi\phi}) = \tag{5.39}$$
$$\psi_{,uL} - \frac{1}{2}\cos^2(L)\left(\psi_{,LL} - 2\tan(L)\,\psi_{,L}\right) + \frac{1}{2H(u)}\frac{dH(u)}{du}\psi_{,L} = 0,$$

$$a_{,uL} - \frac{1}{2}\cos^2(L)\,a_{,LL} + \cos^2(L)\tan(L)\,a_{,L} + \psi_{,L}\psi_{,u} = 0, \tag{5.40}$$

and

$$R_{LL} = -2\cos^4(L)\,\psi_{,L}^2. \tag{5.41}$$

Again, for $R_{LL} = 0$ to always be satisfied it must be that $\psi = g(u)$. This also satisfies equations (5.40). Inserting $\psi = g(u)$ into equation (5.41) yields

$$a_{,uL} - \frac{1}{2}\cos^2(L)\, a_{,LL} + \cos^2(L)\tan(L)\, a_{,L} = 0.$$

This is the same as equation (5.37). Hence, for choice $R = H(u)$, equations (5.15), (5.17), and (5.10(c)) are satisfied if $\psi = g(u)$, and $a = F(u) + \int F_2(-(u + 2\tan(L)))e^N dL$.

5.6 Conclusion

The scalar field $\Phi(u,\ r)$ imitates the amplitude behavior of a light ray, or a gravitational wave, if the area of a spacetime slice of constant ϕ and constant z is proportional to $R =$ constant, or $R = H(u)$. For an observer in a spacetime slice of constant ϕ and z, these choices for R correspond to the spacetime slice having a constant area along each null hypersurface u. This also forces the gravitational waves ψ to have the form $\psi = g(u)$. If R is something else besides $R =$ constant or $R = H(u)$, then the pure time derivative term of Φ in equation (5.30) remains and the scalar field no longer has solutions of the form $F(u)$. Other choices of R cause the scalar field to depend on L and the area along each null hypersurface u to change in some manner. An extension of this work would explore the incoming solution of equation (5.30) for various choices of $H(u)$.

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